



# TECHNICAL NOTE

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## A GENERAL DIGITAL COMPUTER ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES

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SUMMARY

The application of high speed digital computers in the rational analysis of statically indeterminate structures, and the significance of this application in airframe design, are discussed.

The matrix formulation of the force method of analysis is reviewed, and the programs which have been produced to generate the matrices and solve the equilibrium and continuity equations are described. These programs are general enough to apply to any linear discrete structure.

Numerous comparisons between analysis and experimental results are presented. In addition, applications of the programs in the production stress analysis of a large commercial jet transport are described. Applications to thermal stress problems and low aspect ratio wings are also included.

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<sup>1</sup>This paper, which carried a Douglas Aircraft Company designation of "Engineering Paper No. 834," was presented before a meeting of the Structures and Materials Panel of the Advisory Group for Aeronautical Research and Development, North Atlantic Treaty Organization, in Aachen, Germany, September 17, 1959. Since the proceedings of the above Panel meeting are not being published, arrangements have been made with AGARD and the Douglas Aircraft Company for the release of this paper in its original form by NASA to increase its availability.

## NOTATION

In the following definitions, the term "analysis condition" means any combination of external load, thermal deformation, support displacement, etc., tending to produce stress and deflection in the structure. The matrices are defined in the order of their appearance in the analysis. Matrices which are not in the list are defined in the text.

<u>Matrix</u>	<u>Definition of the Matrix Element</u>
$Q = [Q_{ij}]$	$Q_{ij}$ = the $i$ th principal statically determinate force resulting from the redundants and the external loads in the $j$ th analysis condition.
$Q_s = [Q_{sij}]$	$Q_{sij}$ = the $i$ th subordinate statically determinate force resulting from the redundants and the external loads in the $j$ th analysis condition.
$X = [X_{ij}]$	$X_{ij}$ = the $i$ th principal redundant in the $j$ th analysis condition.
$X_s = [X_{sij}]$	$X_{sij}$ = the $i$ th subordinate redundant in the $j$ th analysis condition.
$\phi = [\phi_{ij}]$	$\phi_{ij}$ = the $i$ th principal external load in the $j$ th analysis condition.
$\phi_s = [\phi_{sij}]$	$\phi_{sij}$ = the $i$ th subordinate external load in the $j$ th analysis condition.
$m_{pp} = [m_{ppij}]$	$m_{ppij}$ = the component in the $i$ th principal degree of freedom of a unit value of the $j$ th principal statically determinate force.

## Matrix

## Definition of the Matrix Element

$$m_{ps} = m_{ps_{1j}}$$

$m_{ps_{1j}}$  = the component in the  $i$ th principal degree of freedom of a unit value of the  $j$ th subordinate statically determinate force.

$$p_{x_{pp}} = [p_{x_{pp_{1j}}}]$$

$p_{x_{pp_{1j}}}$  = the component in the  $i$ th principal degree of freedom of a unit value of the  $j$ th principal redundant.

$$p_{x_{ps}} = [p_{x_{ps_{1j}}}]$$

$p_{x_{ps_{1j}}}$  = the component in the  $i$ th principal degree of freedom of a unit value of the  $j$ th subordinate redundant.

$$p_{o_{pp}} = [p_{o_{pp_{1j}}}]$$

$p_{o_{pp_{1j}}}$  = the component in the  $i$ th principal degree of freedom of a unit value of the  $j$ th principal external load.

$$p_{o_{ps}} = [p_{o_{ps_{1j}}}]$$

$p_{o_{ps_{1j}}}$  = the component in the  $i$ th principal degree of freedom of a unit value of the  $j$ th subordinate external load.

$$m_{sp} = [m_{sp_{1j}}]$$

$m_{sp_{1j}}$  = the component in the  $i$ th subordinate statically determinate degree of freedom of a unit value of the  $j$ th principal statically determinate force.

$$m_{ss} = [m_{ss_{1j}}]$$

$m_{ss_{1j}}$  = the component in the  $i$ th subordinate statically determinate degree of freedom of a unit value of the  $j$ th subordinate statically determinate force.

$$p_{x_{sp}} = [p_{x_{sp_{1j}}}]$$

$p_{x_{sp_{1j}}}$  = the component in the  $i$ th subordinate redundant degree of freedom of a unit value of the  $j$ th principal redundant.

$$p_{x_{ss}} = [p_{x_{ss_{1j}}}]$$

$p_{x_{ss_{1j}}}$  = the component in the  $i$ th subordinate redundant degree of freedom of a unit value of the  $j$ th subordinate redundant.

$$p_{o_{sp}} = [p_{o_{sp_{1j}}}]$$

$p_{o_{sp_{1j}}}$  = the component in the  $i$ th subordinate external load degree of freedom of a unit value of the  $j$ th principal external load.

Matrix

$$P_{oss} = \begin{bmatrix} p_{oss1j} \end{bmatrix}$$

Definition of the Matrix Element

$p_{oss1j}$  = the component in the  $i$ th subordinate external load degree of freedom of a unit value of the  $j$ th subordinate external load.

## INTRODUCTION

For many years, elementary methods of stress analysis were used almost exclusively in the design of aircraft structures. These methods involved a number of assumptions, including especially the assumptions that plane sections of elongated members remained plane under the action of bending loads, and that, in torque, sections were free to warp. In many parts of the airframe these assumptions were, and are, completely justified by the nature of the structure and the loading. In other places, the assumptions did not apply, as at the roots of wings, or in the regions of fuselage cutouts. In such areas, other assumptions, conservative and often overlapping to ensure safety, were made. Occasionally a more precise analysis was performed, but such occasions were rare.

Actually no other recourse was possible, because the extensive use of precise methods required computing facilities which did not exist. Such facilities, however, are now available. To appreciate the advance which has been made in the art of computation, consider the fact that about twenty seconds are required to multiply two seven digit numbers on a desk calculator, whereas a large automatic computer can multiply 10,000 pairs of such numbers per second. These figures represent an increase in computing power on the order of 200,000 : 1. On a cost basis, the expense of computing has decreased on the order of 5,000 : 1.

The introduction of matrix algebra into structural analysis has facilitated calculations also, by converting what was formerly a complicated mathematical problem into a systematic procedure.

The result of these improvements is that the use of advanced methods in stress analysis is now a practical undertaking. The question is, to what extent should these methods be applied.

Figure 1 shows the results of a test run at NASA on a cylindrical shell supported at one end on a rigid foundation, reinforced by circular rings, and carrying a radial load at the free end. The figure shows the longitudinal tensile and compressive stresses in the shell, as determined from test, as computed by elementary theory ( $My/I$ ), and as computed by rigorous methods. The figure shows that

the maximum bending stress at station 45 frame as computed by elementary theory is in error by a ratio of almost 3.6 to 1, whereas the error resulting from the rigorous computation is only 10%. Notice also that a secondary maximum occurs at the so called "neutral axis" where the stress is supposed to be zero. Even at the rigid support, where the section is forced to remain plane, the error in  $M_y/I$  is still 2.2 to 1. This structure is not an isolated case; it is typical of many parts of the airframe, and there are places in actual structure where errors resulting from elementary analysis may be larger, because of the existence of cutouts or other conditions.

The results of Figure 1 are well confirmed, inasmuch as they were obtained independently by Jensen of the Gruman Aircraft Company and published by him in reference 5. These results cannot be ignored or dismissed; they are facts, and must be considered in any assessment of structural analysis methods.

What is the significance of the errors involved in the use of elementary methods?

Structure analyzed by rough methods and not thoroughly checked by a careful testing program can contain large stress concentrations. These concentrations can produce metal fatigue and cause the structure to have a short life. Much importance has been attached, justifiably, to the effects of small scale stress concentrations around bolt holes, tool marks, small radius fillets, etc., in reducing fatigue life. Perhaps not enough emphasis has been given to the importance of large scale stress concentrations that are not revealed by rough analysis methods. Obviously, an unconservative error of 3 : 1 or more in the computed stress, if undetected, must lead to a short lived structure. In such a case no amount of attention to design details, important as they are, can produce a fatigue resistant component. The possibility exists that many of the fatigue troubles experienced in the operation of present day aircraft have resulted from the use of elementary stress analysis methods where they did not apply.

These large scale stress concentrations can also cause failure under the action of a single load, even though yielding tends to alleviate the condition. The consequences of such a failure need not be emphasized.

If, as is normally the case, a thorough testing program is undertaken, then



all stress concentrations of importance can be discovered and eliminated. However the cost of building, instrumenting, and testing full scale components is very high, even compared to the rental of a large computer. This testing expense continually increases as the demand for higher performance vehicles requires the working of metals to higher operating stresses, the use of unusual configurations, and the ability to withstand severe environmental conditions. The testing of large components and entire airframes at high temperature will be an especially expensive procedure, because of the large power requirements to heat, as well as to cool, the specimen; the complicated apparatus needed for temperature control; the specialized instrumentation, such as high temperature strain gauges required for measurements; and the additional engineering required to plan the test. The new methods of stress analysis can play a very important part in helping to keep these testing expenditures within reasonable limits.

Finally, the financial risk involved in a large aircraft project is sufficient to warrant a double check through both test and accurate analysis to make sure that no defective conditions exist.

The conclusion is drawn, therefore, that the extensive use of advanced digital methods of stress analysis is justified at the present time, and that these methods will become even more important in the future.

#### SCOPE OF THE PAPER

The paper contains a general description of the method and sections on the matrix formulation, computer programs, analysis procedures, comparisons with test results, and applications. For a non-technical description of the work, the sections on the method, test results, and applications are recommended.

#### ACKNOWLEDGEMENTS

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## THE METHOD

In the following discussion, the term "discrete structure" denotes a structure composed of a finite number of members connected at a finite number of joints. The term "linear structure" denotes a structure for which the relationships between external load, support displacement, internal force, and deflection are linear.

Almost every procedure for the analysis of statically indeterminate structures can be classified as either a "force" or a "displacement" method. In the force method, the unknown internal forces are calculated first; the displacements second. In the displacement method, the displacements are calculated before the forces. Argyris [1]\* has discussed the two methods and shown the existence of an analogy between them.

The capabilities of the digital computer allow either of the basic methods to be programmed in its simplest and most general form. In the past, a great many variations of the basic methods have been employed. One reason for such diversity has been the need to avoid extensive calculation by tailoring the method to fit the structure. However, the development of the digital computer has altered the situation. Extensive calculations now can be performed rapidly and economically. Therefore, a return to basic principles is feasible and, furthermore, the computer program designed to utilize these principles can be general in its applications.

Some of the advantages to be gained from a basic, general approach are reduced programming time, reduced training of personnel, the added insight that results from the application of basic principles, and the reduction of errors that results from familiarization in the use of a single method.

The method of analysis described in this paper is a matrix formulation of the equilibrium equations and the Maxwell-Mohr equations for statically indeterminate structures. This formulation was presented at a meeting of the Second U.S. Congress of Applied Mechanics in June, 1954 [2]. The use of matrix algebra is now recognized as essential in preparing the structural analysis problem for the computer. Langefors [3] and Wehle and Lansing [4] had previously published

\* Numerals in brackets indicate references.

matrix formulations of Castigliano's Theorem. However, the Maxwell-Mohr equations are a little simpler in form because they do not involve partial derivatives. Also, the applications to thermal stress and nonlinear problems are more straight-forward.

In the Maxwell-Mohr method, which is a force method, the structure is cut to create a statically determinate structure or basic system. The members of the statically determinate structure may be simple elements, or they may themselves be complicated statically indeterminate structures. (In fact, even so called simple elements are actually infinitely redundant). After cutting, values of the redundants are chosen such that the deflections at the cuts resulting from external loads, support displacements, element thermal and other deformations, and from the redundants, are zero. The redundants can be either forces existing at the cuts, or linearly independent combinations of these forces, as Argyris has pointed out [1]. The conditioning of the simultaneous equations involved in solving for the redundants can be improved either by cutting on the basis of physical reasoning so that the forces at the cuts are small compared to other forces in the structure, or by linearly transforming the redundants on the basis of the known orthogonal solution of a geometrically regular structure which bears a resemblance to the structure under consideration. The use of statically indeterminate substructures as elements, which have been previously analyzed, also improves the conditioning.

The present method comprising the equilibrium and Maxwell-Mohr equations and the associated digital computer program is applicable to any linear discrete structure, and through iterative techniques to certain nonlinear structures as well. The method applies not only to various parts of the airframe structure such as the wing-fuselage intersection, the tail-fuselage intersection, the cockpit enclosure, the area surrounding a fuselage cut-out, a low aspect ratio wing, and so on, but also to many types of structures encountered in civil engineering practice.

This generality was not designed into the method to show the versatility of the computer, but because generality is necessary if the analyst is to have the tools that he needs to deal with the problems arising in airframe and missile design. Thus, many important airframe components have no recognizable geometric

regularity such as would permit the use of simplifying but restrictive assumptions, or the application of results from elasticity theory. Figures 2a and 2b, which show a pylon-wing intersection, illustrate a structure of this kind.

### MATRIX FORMULATION

The matrix formulation is preceded by a set of equations in vector notation which permit the calculation of the elements of the equilibrium matrices.

Equilibrium equations for a statically determinate structure are written by setting the sum of components of forces in a given direction and the sum of moments about a given axis equal to zero. In general, such a set of equations can be expressed in matrix notation in the form  $MQ + P\phi = 0$ . In this equation,  $Q$  is a matrix of unknown generalized forces where the term "generalized force" is understood to mean either a force or a moment. The coefficients of the unknown forces  $Q$  are contained in  $M$ . These coefficients, called generalized components, are force or moment components in certain directions or about certain axes of unit values of the generalized forces.

The matrix  $\phi$  is a matrix of external loads acting on the structure, while  $P$  contains generalized components of unit values of these external loads.

The structure to be analyzed is broken into free bodies, and equilibrium equations are written for each body. The equations are numbered consecutively beginning with one, and to each equilibrium equation there is assigned a correspondingly numbered unit vector coinciding with the direction in which forces are summed or about which moments are taken. These vectors are called degree of freedom vectors, because only as many of them may be assigned to a free body as the body has degrees of freedom if the corresponding equations are to be independent. Figure 3 shows a free body diagram with forces and degree of freedom vectors representing equations of equilibrium. Degree of freedom vectors are shown dotted.

The existence of two kinds of equilibrium equations and two kinds of generalized forces means that there can be four kinds of generalized components. Equations 1, 2, 3, and 4 of Table 1 provide the method for calculating these quantities. In these equations,  $T_j$  is a unit degree of freedom vector (either translational or rotational), and  $F_j$  is a unit generalized force (either a force or a

moment). The symbol  $m_{1j}$  denotes the corresponding generalized component. In the rotation-force equation,  $r_1$  is a vector joining the origin to any point on the line of action of  $T_1$ , and  $\bar{r}_j$  is a similar vector joining the origin to any point on the line of action of  $F_j$ . In equations (1) to (4), the frame of reference is assumed to be a right-handed rectangular Cartesian coordinate system, and rotations and moments are represented by vectors according to the right-hand rule.

After the statically indeterminate structure is cut, three kinds of forces are seen to be acting upon, or in, the determinate structure. These forces are the external loads, the redundants, and the unknown internal forces, referred to

TABLE 1

SUMMARY OF EQUATIONS

GENERALIZED FORCE COMPONENTS

$$\text{Translation-force} \quad m_{1j} = T_1 \cdot F_j \quad (1)$$

$$\text{Rotation-force} \quad m_{1j} = T_1 \cdot \left[ (\bar{r}_j - r_1) \times F_j \right] \quad (2)$$

$$\text{Translation-moment} \quad m_{1j} = 0 \quad (3)$$

$$\text{Rotation-moment} \quad m_{1j} = T_1 \cdot F_j \quad (4)$$

THE K TRANSFORMATION MATRICES

$$K_m = -m_{ss}^{-1} m_{sp} \quad (5)$$

$$K_x = -p_{xss}^{-1} p_{xsp} \quad (6)$$

$$K_o = -p_{oss}^{-1} p_{osp} \quad (7)$$

COEFFICIENT MATRICES IN THE PRINCIPAL EQUILIBRIUM EQUATION

$$M = m_{pp} + m_{ps} K_m \quad (8)$$

$$P_x = p_{x_{pp}} + p_{x_{ps}} K_s \quad (9)$$

$$P_o = p_{o_{pp}} + p_{o_{ps}} K_o \quad (10)$$

STATICALLY DETERMINATE FORCES RESULTING FROM UNIT REDUNDANTS AND UNIT EXTERNAL LOADS

$$q_x = -M^{-1} P_x \quad (11)$$

$$q_o = -M^{-1} P_o \quad (12)$$

ELEMENT FORCE AND STATICALLY DETERMINATE REACTION MATRICES

$$f_x = Nq_x + H_x \quad (13)$$

$$r_{Dx} = N_r q_x \quad (16)$$

$$f_o = Nq_o + H_o \quad (14)$$

$$r_{Do} = N_r q_o \quad (17)$$

$$f_\Delta = f_o c_\Delta \quad (15)$$

$$r_{D\Delta} = r_{Do} c_\Delta \quad (18)$$

REDUNDANTS

$$\delta_{xx} = f_x^T Df_x \quad (19)$$

$$\delta_{xT} = f_x^T e_T \quad (21)$$

$$\delta_{xo} = f_x^T (Df_o + D_{Fo}) \quad (20)$$

$$\delta_{xR} = \Delta_x + r_{Dx}^T \Delta_D \quad (22)$$

$$X = -\delta_{xx}^{-1} (\delta_{xo} \phi + \delta_{xT} - \delta_{xR}) \quad (23)$$

ELEMENT FORCES AND STATICALLY DETERMINATE FORCES

$$F = f_x X + f_o \phi \quad (24)$$

$$Q = q_x X + q_o \phi \quad (25)$$

DEFLECTIONS

$$\Delta = (f_\Delta^T D + D_{\Delta F})F + (f_\Delta^T D_{Fo} + D_{\Delta o}) \phi + (f_\Delta^T e_T + e_{\Delta T}) - r_{D\Delta}^T \Delta_D \quad (26)$$

or

$$\Delta = (F_{\Delta}^T D + D_{\Delta F})F + (F_{\Delta}^T D_{Fo} + D_{\Delta o})\phi + (F_{\Delta}^T e_T + e_{\Delta T}) - R_{D\Delta}^T \Delta_D - X_{\Delta}^T \Delta_X \quad (27)$$

hereafter as statically determinate forces. The redundants are also unknown, of course, but the statically determinate forces resulting from unit values of the redundants are calculated, and these results are used in the continuity analysis. Calculating the statically determinate forces resulting from unit values of the external loads is also expedient.

Each of the three sets of forces - external, redundant, and statically determinate - is further divided into two subsets called principal and subordinate forces. The subordinates are forces which can be expressed in terms of the principals by a preliminary calculation performed on the machine, after which the subordinate forces are eliminated from the problem. The principal forces are the forces that remain. The purpose of this elimination is to conserve machine capacity.

The choice of subordinates should be such that they can be expressed easily in terms of their principals. For example, consider the shear panel of Figure 4. The forces on this panel form a self-contained system, and any three can be written in terms of the fourth. Thus  $Q_{s1} = Q_1 a/b$ ,  $Q_{s2} = Q_1$ , and  $Q_{s3} = Q_1 a/b$ . The force  $Q_1$  is the principal, and  $Q_{s1}$ ,  $Q_{s2}$ , and  $Q_{s3}$  are subordinates. By this device often half of the forces can be eliminated from the problem.

The next step in the analysis, then, is to designate and number consecutively, beginning with one, each of the following six sets of forces: principal and subordinate statically determinate forces, redundants, and external loads. Matrices of these forces are denoted respectively by the symbols  $Q$ ,  $Q_s$ ,  $X$ ,  $X_s$ ,  $\phi$ , and  $\phi_s$ . Figure 3 illustrates a typical free body diagram with the forces numbered. On this diagram, only statically determinate forces are shown. Redundants and external loads are shown on separate sheets to avoid confusion. Principal force numbers are enclosed in parentheses; subordinate force numbers are not.

After the principal and subordinate forces are chosen, so-called subordinate degree of freedom vectors corresponding to equations of equilibrium are assigned, so that the subordinate forces can be calculated in terms of their principals.

These vectors are shown in the figure by dotted arrows with index numbers not enclosed in parentheses. Finally, principal degree of freedom vectors are assigned to permit the calculation of the principal statically determinate forces. The principal degree of freedom vectors are indicated by dotted arrows with index numbers enclosed in parentheses. In general, four sets of degree of freedom vectors are assigned as follows: principal degrees of freedom, and subordinate statically determinate, redundant, and external load degrees of freedom.

The equilibrium equations can now be written, in matrix notation, in terms of the six sets of forces acting on the free bodies, as follows:

$$\begin{bmatrix} m_{pp} & m_{ps} & p_{x_{pp}} & p_{x_{ps}} & p_{o_{pp}} & p_{o_{ps}} \\ m_{sp} & m_{ss} & & & & \\ & & p_{x_{sp}} & p_{x_{ss}} & & \\ & & & & p_{o_{sp}} & p_{o_{ss}} \end{bmatrix} \begin{bmatrix} Q \\ Q_s \\ X \\ X_s \\ \phi \\ \phi_s \end{bmatrix} = 0 \quad (28)$$

The forces acting on the free bodies are contained in the post multiplier; the generalized components are contained in the premultiplier. The significance of the partitions  $m_{pp}$ ,  $m_{ps}$ , etc., is given in detail in the table of notation. All of the generalized components are computed by equations 1, 2, 3, and 4. The null partitions in the generalized component matrix result from choosing subordinate forces in such a way that they always form small self-contained systems with their principals.

Equation 28 is expanded as follows:

$$m_{pp} Q + m_{ps} Q_s + p_{x_{pp}} X + p_{x_{ps}} X_s + p_{o_{pp}} \phi + p_{o_{ps}} \phi_s = 0 \quad (29)$$



$$m_{sp} Q + m_{ss} Q_s = 0$$

$$p_{x_{sp}} X + p_{x_{ss}} X_s = 0$$

$$p_{o_{sp}} \phi + p_{o_{ss}} \phi_s = 0$$

$$\therefore Q_s = -m_{ss}^{-1} m_{sp} Q$$

$$X_s = -p_{x_{ss}}^{-1} p_{x_{sp}} X$$

$$\phi_s = -p_{o_{ss}}^{-1} p_{o_{sp}} \phi$$

The matrices  $K_m$ ,  $K_x$ , and  $K_o$  are now defined according to equations 5, 6, and 7 of Table 1.

$$\therefore Q_s = K_m Q$$

$$X_s = K_x X$$

$$\phi_s = K_o \phi$$

Substituting these expressions into equation (29) gives

$$MQ + P_x X + P_o \phi = 0, \quad (30)$$

where the matrices  $M$ ,  $P_x$ , and  $P_o$  are defined by equations 8, 9, and 10 of Table 1. Equation (30) is the principal equilibrium equation.

Notice that the matrices  $m_{ss}$ ,  $p_{x_{ss}}$ , and  $p_{o_{ss}}$ , appearing in equations 5, 6, and 7, must be nonsingular. This nonsingularity is obtained by proper choice of subordinate degree of freedom vectors. As a matter of computing convenience, the choice of these vectors should be such that the matrices  $m_{ss}$ ,  $p_{x_{ss}}$ , and  $p_{o_{ss}}$  are lower triangular, because in this event a very rapid computing program can be used to solve the equations. Such a choice is always easy to make, and it has the additional advantage that a lower triangular matrix with nonzero elements everywhere

on the diagonal is nonsingular, and well-conditioned.

Taking  $X = I$  (the unit matrix) and  $\phi = 0$  (the null matrix) in equation (30) leads to equation (11) of Table 1, where  $q_x$  is a matrix of statically determinate forces resulting from unit values of the redundants.

Taking  $X = 0$  and  $\phi = I$  leads to equation (12), where  $q_0$  is a matrix of statically determinate forces resulting from unit values of the external loads.

Check degree of freedom vectors are assigned to various free bodies of the structure so that additional check equations are generated. Such equations provide reliable verification of the calculations up to this stage.

After the equilibrium problem is solved and checked, two additional operations are performed, before the continuity of the structure is restored. First, all of the statically determinate forces, the redundants, and perhaps some of the external loads, are grouped into a single set of forces, called element forces, to facilitate calculating deflections. Second, the statically determinate reactions are grouped into a separate matrix, to permit calculating the effect of support displacements.

Element forces are defined in the following way: Consider any element of the structure which is capable of undergoing deformation, and therefore of contributing to the deflection of the structure as a whole. Both internal forces and external loads may act upon such an element, since the possibility of external loads acting between joints is not excluded. Certain forces acting on the element are designated as element reactions. These element reactions may be internal forces or fictitious forces, but they must be chosen in such a way that they are capable of balancing the other forces applied to the element. The remaining internal forces are designated as element forces. After element forces for the entire structure are selected, they are numbered consecutively beginning with one.

For each element force there is a corresponding element deformation. An element deformation is defined as the component of the displacement of an element force, in the direction of the element force, when the element reactions are undisplaced parallel to themselves.

Figure 5 shows a bending element, with element reactions (indicated thus:  $\rightarrow$ ), element forces ( $F_1, F_2, F_3$ ), and element deformations ( $e_1, e_2, e_3$ ). Other choices of element reactions, forces, and deformations are possible for such an element.

The element deformations are given the same index numbers as the corresponding element forces; and a deformation is positive when it has the same direction as a positive value of the corresponding force. The sign convention for element forces is arbitrary, except that the choice of a sign convention which results in negative off-diagonal flexibility factors (defined later) is not advisable.

Some of the element forces correspond to statically determinate forces; others correspond to redundants and a few may correspond to external loads. Therefore, the element forces can be written in terms of the statically determinate forces, the redundants, and the external loads, as follows:

$$F = NQ + H_X X + H_O \phi, \quad (31)$$

where  $F$  is a matrix of element forces.

If the element forces have been chosen in such a way that each one corresponds exactly to a statically determinate force, a redundant, or an external load, and such a choice should be made, then the matrices  $N$ ,  $H_X$ , and  $H_O$  contain 1's and 0's, and there will be no more than one 1 in any row or column. Such matrices are called extractors, because their only function is to extract information from other matrices.

Setting  $X = I$  and  $\phi = 0$  in equation (31) yields equation (13) of Table 1, where  $f_X$  is a matrix of element forces resulting from unit values of the redundants. Setting  $X = 0$  and  $\phi = I$  yields equation (14), where  $f_O$  is a matrix of element forces resulting from unit values of the external loads.

In the Maxwell-Mohr method, deflections are calculated by applying unit dummy loads coinciding in position and direction with the desired deflections. In the present formulation the assumption is made that a unit external load is applied to coincide with every such deflection. Therefore, a matrix  $f_\Delta$  can be extracted from  $f_O$ , as in equation (15), where  $f_\Delta$  is a matrix of element forces

resulting from unit values of the dummy deflection loads, and  $C_{\Delta}$  is a suitable extractor matrix.

Number the statically determinate reactions consecutively beginning with 1. Then the statically determinate reaction matrix  $R_D$  can be extracted from the statically determinate force matrix as follows:

$$R_D = N_r Q ,$$

where  $N_r$  is a suitable extractor. Setting  $X$  and  $\phi$  equal to  $I$  and  $0$  in turn leads to equations (16) and (17), where  $r_{Dx}$  and  $r_{Do}$  are matrices of the statically determinate reactions resulting from unit values of the redundants and external loads respectively. A matrix  $r_{D\Delta}$  of statically determinate reactions resulting from unit values of the dummy deflection loads is extracted from  $r_{Do}$  as in equation (18).

The essentials of the derivation of equations (19) to (26), inclusive, have been given in reference 2. A feature of this derivation is that although it is based on the conservation of energy, it does not involve elastic strain energy, so that the deflection equations are immediately valid for arbitrary element deformations, including deformations resulting from thermal gradients, plasticity, creep, etc. The derivation is also facilitated by the use of the notions of element reactions, forces, and deformations, as defined above. However, the equations have been generalized to include the effects of support displacements, the application of external loads between joints, and the calculation of deflections at points between joints.

The symbol  $D$  appearing in these equations denotes the flexibility matrix. The elements of this matrix represent element deformations resulting from unit values of element forces. For example, the flexibility coefficients for the beam element of Figure 5 are as follows, if shear deformations are not considered:

$$D_{11} = L/AE, \quad D_{22} = L^3/3EI, \quad D_{23} = D_{32} = L^2/2EI, \quad D_{33} = L/EI,$$

where  $L$ ,  $A$ ,  $I$ , and  $E$  are the length, area, moment of inertia and modulus of elasticity of the member.

The matrix  $D_{Fo}$  contains element deformations resulting from external loads

applied directly to the elements. If loads are applied only at joints, then  $D_{Fo}$  is null. Figure 6 shows the element of Figure 5, with an intermediate load.

The following elements of the  $D_{Fo}$  matrix can be derived by elementary methods:

$$D_{Fo_{1j}} = a \cos \alpha / AE, \quad D_{Fo_{2j}} = a^2 (L-a/3) \sin \alpha / 2EI, \quad D_{Fo_{3j}} = a^2 \sin \alpha / 2EI.$$

The matrix  $D_{\Delta F}$  contains displacements of dummy deflection loads acting directly upon the element, resulting from unit values of the element forces, when the element reactions are not displaced parallel to themselves. Figure 7 shows the element of Figure 5 with an intermediate dummy deflection load. The elements of  $D_{\Delta F}$  are as follows:

$$D_{\Delta F_{j1}} = b \cos \beta / AE, \quad D_{\Delta F_{j2}} = b^2 (L-b/3) \sin \beta / 2EI, \quad D_{\Delta F_{j3}} = b^2 \sin \beta / 2EI.$$

The matrix  $D_{\Delta o}$  contains displacements of dummy deflection loads acting directly upon the element, resulting from unit external loads acting directly upon the element, when the element reactions are not displaced. Figure 8 shows a bending element subjected to intermediate external and deflection loads. The corresponding element of  $D_{\Delta o}$  is as follows:

$$D_{\Delta o_{1j}} = \frac{b}{EA} \cos \alpha \cos \beta + \frac{b^2(3a-b)}{6EI} \sin \alpha \sin \beta \quad \text{if } a > b$$

or

$$D_{\Delta o_{1j}} = \frac{a}{EA} \cos \alpha \cos \beta + \frac{a^2(3b-a)}{6EI} \sin \alpha \sin \beta \quad \text{if } b > a.$$

The matrix  $e_T$  contains element deformations resulting from heating, plasticity, creep, etc. For example, suppose that the tensile element of Figure 9 (a), has been assigned the  $i$ th element force, as shown. In (b) the temperature of the element is increased an amount  $\Delta T$  in the  $j$ th analysis condition. The thermal deformation is then  $e_{T_{1j}} = \alpha L \Delta T$ , where  $\alpha$  is the coefficient of expansion. The matrix elements  $e_{T_{1j}}$  can also represent bending thermal deformations of bars heated unequally on the two sides, or any other kind of a thermal deformation. When the  $e_{T_1}$  represent plastic or creep deformations, they either must be known, as

they could be in a statically determinate structure, or they must have been computed in a previous cycle of some kind of iterative process.

The matrix  $e_{\Delta T}$  contains displacements of the dummy deflection loads acting directly upon the element, resulting from heating, etc., when the element reactions are not displaced parallel to themselves. Figure 10 shows the element of Figure 9 with an intermediate dummy deflection load. The intermediate thermal deformation is  $e_{\Delta T_{1j}} = \alpha \Delta T$ .

The matrices  $\Delta_D$  and  $\Delta_X$  contain displacements of the statically determinate and redundant reactions, respectively. The elements of these matrices are positive when the corresponding support displacements have the same sense as positive values of the reactions acting upon the structure.

Equation (27) provides an alternate, more accurate, but somewhat more cumbersome means of calculating deflections. In this equation,  $F_{\Delta}$ ,  $X_{\Delta}$ , and  $R_{D\Delta}$  are matrices containing element forces, redundants, and statically determinate reactions, respectively, in the uncut structure resulting from unit values of the dummy deflection loads. The equation can be shown to be mathematically identical to equation (26).

#### COMPUTER PROGRAMS

The calculations are performed on an IBM 709 computer. The only "709" program written specifically for the Maxwell-Mohr method is called "Matrix Generation". This program accepts, as input, coordinates and directions numbers which define the degree of freedom and force vectors appearing on the free body diagrams. The direction numbers have previously been computed from the coordinates by an auxiliary program. Thus, the only numerical input prepared by the analyst for this phase is a table of coordinates. The program then generates the elements of the matrices  $m_{pp}$ ,  $m_{ps}$ ,  $p_{x_{pp}}$ ,  $p_{x_{ps}}$ ,  $p_{o_{pp}}$ ,  $p_{o_{ps}}$ ,  $m_{sp}$ ,  $m_{ss}$ ,  $p_{x_{sp}}$ ,  $p_{x_{ss}}$ ,  $p_{o_{sp}}$ , and  $p_{o_{ss}}$  by means of equations (1) to (4) of Table 1.

All the rest of the calculations, as required by equations (5) to (26), are performed with the aid of a general purpose interpretive routine called the "Tape

Matrix Compiler". This routine essentially permits the analyst to write his own programs for matrix operations. Matrices of member flexibilities, loads, thermal deformations, and support displacements, and certain extractor matrices, are input. The machine outputs the unknown forces and deflections of the structure.

The compiler is also used to perform additional operations not covered by equations (5) to (26). These auxiliary operations can include transforming the redundants to improve conditioning, and the modification of member flexibilities, including the complete removal of members.

The joining of structures to form larger structures is accomplished by the basic program, comprising equations (1) to (26).

A program under development, called the "Structure Cutter", permits the machine to select its own redundants optimized to yield well-conditioned equations. The capabilities of the Structure Cutter are briefly discussed in a later paragraph.

## ANALYSIS PROCEDURES

### IDEALIZING THE STRUCTURE

The actual structure is replaced by an idealized discrete structure consisting usually of bars and panels. In general the bars can carry tension, torque, two components of bending moment, and two components of shear. The panels can carry shear and biaxial tension. In the most generally useful idealization, bars are considered straight between joints, and panels carry only shear. However panels are permitted to be warped. This allowance for panel warping improves the accuracy of the analysis, because joints of the idealization can lie on the true contour of the actual structure. Furthermore, warping simplifies the input, because there are few if any derived coordinates. The meaning of the term "derived coordinates" is explained later.

Panels should be rectangular if possible, trapezoidal if not rectangular, or at least nearly trapezoidal. Panels that almost come to a point should be avoided. Triangular panels should probably be removed, leaving a triangular framework of bars.

A problem of structural idealization concerns the question of the attachment of shear panels to bars. Two methods of attachment are considered. In the first method, panels are attached to bars at the midpoints of panel edges, as shown at "A" of Figure 12. In the second method, the attachment is continuous, as shown at "B", and the assumption is made that load in the adjacent bars varies linearly between joints.

Figure 11 shows a set of skin-stringer panels, rigidly supported at infinity. The panels have symmetry about the X-axis, the stringers are equally spaced and have constant area, all the stringers are equally stiff, and the sheet thickness is constant. Transversely the panels are assumed to be stiffened by a continuum of infinitely rigid bars. Axial loads are applied to the #3 stringers at  $X = 0$ .

The exact solution of the stringer loads and panel shear flows in the structure was obtained. The structure was also analyzed by the Maxwell-Mohr method, for the idealization shown in Figure 12. At  $X = 80$ , conditions are essentially the same as they are at infinity.



Two digital solutions were obtained. In the first solution, panels were assumed to be attached to bars at panel mid-points only. Under this assumption, the load in a bar is constant, but can jump abruptly at joints and panel mid-points. The flexibility matrix corresponding to this assumption is diagonal.

In the second solution the load in the bar is assumed to vary linearly between joints. The flexibility matrix in this case is not diagonal.

The comparison of the three solutions for stringer loads is given in Table 2. The results for methods 1 and 2 are followed by the percent errors in parentheses. The comparison for shear flows is given in Figure 12.

Table 2 Stringer Loads

Stringer Number	X	0	20	40	60	80
1	Exact	0	.1470	.1889	.1987	.1996
	Method 1	0 (0)	.1446 (-2%)	.1869 (-1%)	.1970 (-1%)	.1988 (-0%)
	Method 2	0 (0)	.1344 (-9%)	.1885 (-0%)	.1986 (-0%)	.1996 (0%)
2	Exact	0	.2015	.2030	.2007	.2002
	Method 1	0 (0)	.1925 (-4%)	.2020 (-0%)	.2008 (0%)	.2004 (0%)
	Method 2	0 (0)	.1864 (-8%)	.2060 (1%)	.2005 (-0%)	.2001 (-0%)
3	Exact	1.0000	.3031	.2152	.2029	.2005
	Method 1	1.0000 (0)	.3258 (7%)	.2221 (3%)	.2044 (1%)	.2017 (1%)
	Method 2	1.0000 (0)	.3584 (18%)	.2110 (-2%)	.2018 (-1%)	.2004 (-0%)

The comparisons show that the "panel mid-point method" gives greatest accuracy. The fact should be noted however that this method gives somewhat less accuracy than the second method for the deflection of a cantilever thin web beam, idealized as shown in Figure 14. Here the accuracy of the deflection computed by the first method depends on the number of bays and is satisfactory for four bays. Both methods give correct cap loads and shear flows for any number of bays.

Since the "panel mid-point method" is the simplest, and seems to be the most accurate, at least for stresses, it appears to be preferable to the second method.

A distinction is made between "defining" and "derived" coordinates. This distinction is demonstrated in Figure 15, which shows a pin-jointed truss lying in the X-Y plane. Member AC is assumed straight. The geometry of the truss therefore may be considered to be defined by the X and Y coordinates of points A, B, and C, and the X coordinate of D. The coordinate  $Y_D$  can be derived from  $X_D$  on the assumption that AC is straight. The coordinate  $Y_D$  is therefore a derived coordinate, and the others are defining coordinates. Defining coordinates should be input with an accuracy of about six decimal places to avoid contradictions between them and the assumptions upon which they are derived, within the machine. Because of this accuracy requirement, derived coordinates should be avoided.

A warped shear panel cannot be in equilibrium under the action of shear forces alone, as Figure 16 demonstrates. The shear forces shown in the plan view all have downward components in the edge view. The panel can be put into equilibrium with the addition of two forces at opposite corners, as shown in the perspective view of Figure 17. This figure also shows principal and subordinate force numbers, and subordinate degree of freedom vectors, which can be assigned to permit the machine to calculate the subordinate forces in terms of their principals. The warping forces are approximately normal to the panel. The reactions to the warping forces are assumed to act on joints.

Many structures contain warped panels which cannot be flattened in the idealization without seriously compromising the accuracy of the solution. Furthermore, the flattening process is usually more trouble than accounting for the warping.

#### CUTTING THE STRUCTURE

Box structures, like wings, composed of bars in tension and panels in shear, tend to be better conditioned, because they are stiffer, than fuselage-type structures which contain flexible rings. For structures which are inherently well conditioned, and yet which may offer cutting difficulties because of unusual features, the "building method" is a useful procedure.

In the building method, a unit of the structure known to be statically determinate is selected, and the structure is built from this unit by adding other statically determinate units. The members which are omitted in the process are the redundants.

Figure 18 (a) shows the uncut structure, (b) and (c) show stages in the building method and (d) shows the final cut structure. Two panels and a reaction are redundant. In the process, the use of "temporary reactions" may be expedient. These reactions can be replaced by the actual reactions at the completion of the process.

The following expression is convenient for checking the degree of redundancy of a structure composed of shear panels and axially loaded bars:

$$n = b + p + r - 2j_2 - 3j_3$$

where  $b$  = the number of uncut bars,  
 $p$  = the number of uncut panels,  
 $r$  = the number of reactions,  
 $j_2$  = the number of two constraint joints,  
 $j_3$  = the number of three constraint joints,

For a statically determinate structure,  $n = 0$ . The expression, with  $n = 0$ , is a necessary but not a sufficient condition for static determinacy. For the structure of Figure 18,  $n = 28 + 14 + 6 - 2 \times 0 - 3 \times 16 = 0$ .

#### DIAGRAMS

The following diagrams are utilized: (1) a general view of the idealized structure with the joints numbered consecutively beginning with one, (2) a set of free body diagrams, and (3) diagrams showing the element forces.

The free body diagrams have been described in the section on matrix formulation, and Figure 3 shows a typical diagram for statically determinate forces. The only feature of these diagrams not already mentioned are the free body numbers, shown enclosed in squares in Figure 3. The machine uses these numbers to associate forces with their corresponding degrees of freedom.

The element force diagrams show element reactions and element forces, the latter being numbered consecutively beginning with one. The statically determinate forces and redundants should be chosen so that each element force is identical with either a statically determinate force or a redundant, so that the elements of the  $N$  and  $H_X$  matrices consist only of 1's and 0's.

#### LOAD SHEETS

Data is input on three different formats, as follows: the coordinate table, the vector description tables, and the matrix load sheet.

The coordinate table is a list of joint numbers with their associated X, Y, and Z coordinates. With the aid of an auxiliary program, the machine computes a table of direction cosines of vectors defined by point pairs of the coordinate tables. The point pairs are specified by the analyst on a separate load sheet. The auxiliary program can also compute the direction cosines of a vector defined as the cross product of two other vectors each in turn defined by point pairs designated by the analyst. The vectors for which direction cosines are calculated include most, or all, of the vectors which appear in the analysis. Direction numbers of additional vectors can be hand input if necessary. The machine sorts the computed direction cosines according to the defining points, and assigns each set of X, Y, and Z direction cosines a serial number.

The vector description tables are of two types. On the type 1 table the following information is input for each vector: the vector serial number; the type, whether angular or linear; the sign; the number of the free body upon which the vector acts; the number of a point on the line of action of the vector; and the serial number of the direction of the vector. Each vector is listed only once in the type 1 load sheets. However most of the force vectors appear more than once on the free bodies, and an entry must be made each time a vector appears. These additional entries are made on the type 2 tables which have provision only for vector serial numbers, signs, and free body numbers. The type 1 and type 2 tables are filled out for the four kinds of degree of freedom vectors, and the six kinds of force vectors mentioned previously.

The matrix load sheets contain spaces for the matrix elements, and for the row and column numbers corresponding to each element. The matrices  $N$ ,  $H_X$ ,  $H_O$ ,  $C_\Delta$ ,  $D$ ,  $D_{FO}$ ,  $D_{AO}$ ,  $e_T$ ,  $e_{AT}$ ,  $\Delta_D$ ,  $\Delta_X$  and  $\phi$  are input on these sheets. Occasionally some elements of the  $K_O$  matrix also are hand input.

Ordinarily only the matrices  $N$ ,  $H_X$ ,  $H_O$ ,  $C_\Delta$ ,  $D$  and  $\phi$  are required, and of these matrices  $N$ ,  $H_X$ ,  $H_O$  and  $C_\Delta$  should contain only 1's and 0's. Thus the only formats which contain numerical input are the coordinate table, the flexibility matrix  $D$ , and the load matrix  $\phi$ . Therefore a problem which has been set up for a given set of coordinates, flexibilities, and external load can be solved for new coordinates, flexibilities, and loads by inputting only three tables. These

tables represent the minimum possible input for the problem. Therefore a given set-up, say for a fuselage section, can be used many times for a variety of fuselage analyses, and the set-up essentially becomes, in itself, a general program for fuselage problems.

#### CHECKS ON THE OUTPUT

The equilibrium checks, made by writing extra equations of equilibrium, have been mentioned. Two other important types of checks are the simultaneous equation checks and the symmetry checks. Simultaneous equation checks are made on the solutions of both the equilibrium and the continuity equations by substituting the results into the original equations. A symmetry check is made on  $\delta_{xx}$ , which must be symmetric by Maxwell's law. A similar check is made on the deflection matrix  $\Delta$ , for rows and columns which correspond to identical unit deflection loads and external loads.

#### IMPROVING THE CONDITIONING

Naturally every effort should be made at the beginning to secure well-conditioned equations. The familiar rule is that redundants should be chosen which are small compared to other forces in the structure. The rule can also be stated as follows: in the cutting process the structure should lose as little stiffness as possible. For example, a good choice of redundants for a fuselage frame is the insertion of three hinges. A complete cut at one point leaves the frame very flexible.

A second device is to break the structure into statically indeterminate substructures. The substructures are then cut and analyzed, after which they are joined to form the original structure, as discussed in a later paragraph. At each stage of this process the redundants are relatively few in number, and generally well conditioned.

A third device is the utilization of orthogonal solutions derived from the theory of elasticity for geometrically regular bodies which resemble the structure at hand. This process has been thoroughly discussed by Argyris.

#### JOINING SUBSTRUCTURES

In this process the structure is broken, by cutting redundants, into substructures, which remain joined together by other forces which can be computed

from statics. Thus the cut structure can be regarded as a statically determinate structure consisting of statically indeterminate elements. Figure 19 shows a DC-8 wing-pylon intersection which has been broken into two substructures by this method. The figure shows statically determinate forces only. The other joining forces, which are redundants, are shown on a separate sheet. Figures 2a and 2b show details of the idealized substructures.

After the structure has been cut into substructures, each of the substructures is also cut and analyzed in detail, for unit values of the external loads, which include the joining redundants. In particular the deflections of the substructures, at points where they have been cut apart, are calculated. The analysis of each substructure utilizes the basic program and the equations of Table 1.

After the substructures are analyzed, they are joined to form the original structure by another application of the basic program and equations. In this process free body diagrams, like figure 19, are drawn. Element force diagrams are also prepared. Element reactions for the substructures, considered as elements of the original structure, must be identical with the statically determinate reactions that were utilized in the detail analysis of the substructures. This requirement is necessary because the elements of the flexibility matrices  $D$ ,  $D_{FO}$ ,  $D_{AF}$ , and  $D_{AO}$  are extracted from the deflection matrices  $\Delta$ , calculated for each of the substructures. The extraction is accomplished with the aid of extractor matrices consisting of 1's and 0's and the tape matrix compiler.

#### DISCONNECTING AND FLEXIBILITY MODIFICATION

The technique discussed by Argyris [1], Michielsen and Dijk [13], and Best [14], for modifying flexibilities with the aid of arbitrary element deformations after the redundants have been computed, has two important applications. First the effect of changing the sizes of a few members upon the stress distribution can be determined with a minimum amount of calculation. However the method becomes inefficient when the number of elements to be modified becomes equal to or greater than the number of redundants. In this case a new flexibility matrix should be input. Second, the notion of filling in cut-outs, like fuselage doors, and later removing them, is important, because the process of cutting the structure is greatly simplified when cut-outs are not present, and the equations are likely to be better conditioned. However, more machine capacity is required.

Members can also be removed by making them more flexible, say on the order of 1,000,000 times, than other members of the structure. This approach only works when the forces being reduced to zero are redundants. Otherwise the continuity equations tend to be linearly dependent.

#### THE STRUCTURE CUTTER

A method has been devised for having the machine cut the structure. In this approach no distinction is made between statically determinate and redundant forces when the problem is set up. The number of unknowns in the equilibrium equations generated by the machine then exceeds the number of equations. By a process of selecting columns of the rectangular matrix of coefficients of unknowns in these equations, the machine chooses a well-conditioned square matrix. The unknowns which correspond to the columns of this matrix are the statically determinate forces, and the remaining unknowns are the redundants. The choice of columns is influenced by weighting factors which reflect the stiffness of the members of the structure.

Figure 20a shows a statically indeterminate structure. Figure 20b shows the same structure as it was cut by the machine.

#### SIMPLIFIED INPUT

A new program called the "Redundant Force Method" is being developed. This program is basically the same as the method described previously, but the new method incorporates a number of improvements which eliminate the need for preparing free body diagrams, and reduce the input to a minimum. In effect the machine automatically cuts the structure (utilizing the "Structure Cutter"), breaks the statically determinate structure into free bodies, writes and solves the equations of equilibrium, and writes and solves the equations of continuity. A certain penalty in additional machine time is involved, however the new program is expected to be especially useful in the rapid solution of preliminary design problems for which a rough idealization is satisfactory, and which cannot be solved without a large error by elementary methods.

#### NONLINEAR PROBLEMS

Although this subject is beyond the scope of the present paper, some mention should be made of the applications to the nonlinear problems involved in calculating the effects of plasticity and creep upon the behavior of the structure. The approach

to these problems has been through the use of various step-by-step, or iterative, procedures. In all such procedures the question of convergence is of primary importance, because the rate of convergence can be fast or slow, or the process can be divergent. Rapid convergence is necessary, because a large amount of calculation per cycle is required even for a structure of moderate size.

A method of calculating stresses and deflections in the presence of plasticity is given in reference 6. The method utilizes the rapidly convergent Newton-Raphson procedure for solving nonlinear simultaneous equations. Agreement with test results is demonstrated. Reference 7 presents an approach based on the use of fictitious loads which appears to require a minimum amount of computation per cycle.

A step-by-step application of the Maxwell-Mohr analysis to the creep problem is under development. This work is expected to provide a means of computing the history of stress and deflection of a statically indeterminate structure subjected to time dependent load and thermal inputs.



## COMPARISON WITH TEST RESULTS

Comparisons between analysis and test results obtained at the NASA and during the DC-8 static test have been made. The NASA comparisons were accomplished in the period from June 1956 to September 1957. In all the numerical analysis, the midpoint idealization for shear panels was used.

The comparison for axial stresses measured in the cantilever circular cylinder of Figure 1 has been mentioned. Figure 21 shows the analytical and test results for frame bending moments and skin shear flows in the same cylinder. The results of the Maxwell-Mohr analysis are in very close agreement also with results obtained by the method of Hoff [8], as reported in reference 9.

Figures 22, 23, 24, and 25 show comparisons for a swept box tested at the NASA, and reported in reference 10. The box was of rectangular section and had a total of 32 stringers. In the figures the heavy solid lines indicate idealized stringers and bulkheads, while the dotted lines indicate bars obtained by lumping skin in the chordwise direction. The analysis would not yield satisfactory approximations for shear flows in the covers until these bars were inserted. Poisson's ratio was accounted for in the triangular area at the root. In the bending test, the characteristic peaking of axial stress at the rear spar is correctly predicted, as is the reversal of shear flow in the front spar web.

Figures 26 and 27 show comparisons for cylinders with cutouts subjected to bending and torque respectively. The tests are described in references 11 and 12. As the figures show, more idealized stringers were inserted in the upper side than in the lower, because the cutout at the top perturbs the stress field, and requires finer lumping. Frame flexibility was taken into account. The resulting agreement is excellent. However there is one shear panel at the corner of the cutout which, in the bending case, does not have approximately a uniform shear flow, as assumed. At one edge of this panel the shear flow, not shown in the figure, is considerably higher than the value at the panel center. The only way to cover this concentration without going to a finer lumping is with an empirical factor.

Figure 28 shows a comparison of measured and calculated stresses for a station in the root region of the DC-8 wing. The analysis which yielded the calculated results is discussed in a later section.

## APPLICATIONS

The method has been extensively applied in the analysis of jet transport components; missile parts, including fins and body components; and a supersonic low aspect ratio wing. Many of these analyses included calculations of thermal stress and deflection.

The wing-fuselage intersection was one of the primary problems in the stress analysis of the DC-8. The stress distribution was complicated by the existence of wing sweep, an auxiliary spar, landing gear cutouts in the lower part of the fuselage behind the wing, a keel beam running along the fuselage centerline below the floor, and other details. The problem was approached by first making an analysis of the entire region, including a fairly detailed representation of the fuselage, and a simplified idealization of the wing. From the results of this analysis, reaction forces between wing and fuselage were determined. A detailed wing root analysis was then made, in which these reaction forces were applied.

Figure 29 is a diagram of the idealized structure used in the detailed wing analysis, showing the three spar construction, with the auxiliary spar which supports the main landing gear. The idealization had the correct sweep; dihedral; incidence and taper, both in plan-form and in thickness; and the airfoil sections were accurate. However, twist was removed to flatten skin panels. There were 113 redundants and 300 element forces. The first complete calculation based on this idealization was finished in March 1956. Had the job been done a little later, panel warping and twist would have been considered.

The idealized structure for the tail-fuselage intersection is shown in figure 30. The idealization included a portion of the vertical tail, and a stub of the all-movable horizontal surface. Some of the sections were stiffened by frames like the one shown in section A-A; others had partial bulkheads. The joints of the idealized structure lay on the true contour, and panel warping was accounted for. The forward and aft parts of the structure were analyzed separately and then joined at section A-A. The first complete calculation was finished in September 1957.

Deflection influence coefficients calculated for both the wing and the fuselage tail section were used in flutter analysis.

An analysis of the fuselage nose section, including the cockpit enclosure, was performed. The problem was complicated by the presence of cabin pressure, and the fact that the pressure envelope was irregular because of the existence of the unpressurized nose-wheel well below the floor. The members of the cockpit canopy also caused added difficulties, because some of them were designed to carry tension, bending moments about two axes, and torque, and they were so analyzed. The structure was analyzed in two separate sections, which were then joined. After joining, the technique of virtual disconnecting loads was employed to calculate the effect of door cutouts.

Figure 2a and 2b show the idealized structure for the Conway outboard pylon. The structure incorporates a bottoming strut, shown in figure 2a. The bottoming of this strut, after a certain amount of load has been applied, changes the stress distribution, and causes a nonlinearity, which was taken into account.

Figure 31 shows the structure of the JT-4 ejector-reverser. The structure is irregular; has large cutouts for the reversing buckets; incorporates members subjected to tension, bending about two axes, and torque; and is subjected to large thermal gradients. The JT-3 and Conway ejectors are similar. Results from the JT-3 analysis became available within a period of two months. The same set-up was then utilized in the analysis of the JT-4 and Conway ejectors, which have different sizes, shapes, and stiffnesses. The Conway ejector analysis was completed in final form ready for submission to the FAA in one month's time. Spring constants for the ejectors were calculated and shown in proof test to be correct within the experimental error.

Numerous applications to low aspect ratio wing and missile structures have been made, but these projects are classified and cannot be discussed. However the foregoing applications and experimental verifications have demonstrated that the matrix equations and the computer program are sufficiently general to deal with any linear discrete structure. Missile and supersonic airplane structures are no exceptions. Thus the low aspect ratio multi-spar wing-fuselage structure of figure 32 can be analyzed, with all the detail shown and more, with joints on the true contour, for load and thermal stress. Deflections, and a deflection influence matrix useful in flutter analysis also can be output.

## CONCLUSION

A procedure for structural analysis, comprising a matrix formulation of the equilibrium and Maxwell-Mohr continuity equations, and an associated digital computer program, has been developed. This procedure is applicable, in its basic form, to any linear discrete structure. The method has been fully verified by comparison with test results, both in the laboratory and in proof test, and it has been shown to be a practical analysis tool in numerous applications.

Procedures of this kind, several of which have appeared in the last few years, represent a break-through in the art of stress analysis. These methods permit the practical calculation of stresses in complicated shell structures in rigorous accord with basic physical principals. This rigor is necessary, because approximate methods widely used in the past can be in error by large amounts. These errors are alleviated somewhat by stress redistribution above the yield, but below the yield they represent stress concentrations which cause premature fatigue failures. Above the yield premature static failures can occur in spite of the redistribution.

In the past, serious consequences of these errors have been avoided by extensive testing. Some testing will always be necessary, but it is expensive, even compared to the cost of operating a large digital computer. In the future, testing expense will increase as airframes become larger, and the additional complication of thermal gradients is introduced. Therefore the need for rigorous methods is increasing.

Douglas Aircraft Company, Inc.,  
Santa Monica, Calif., September 17, 1959.

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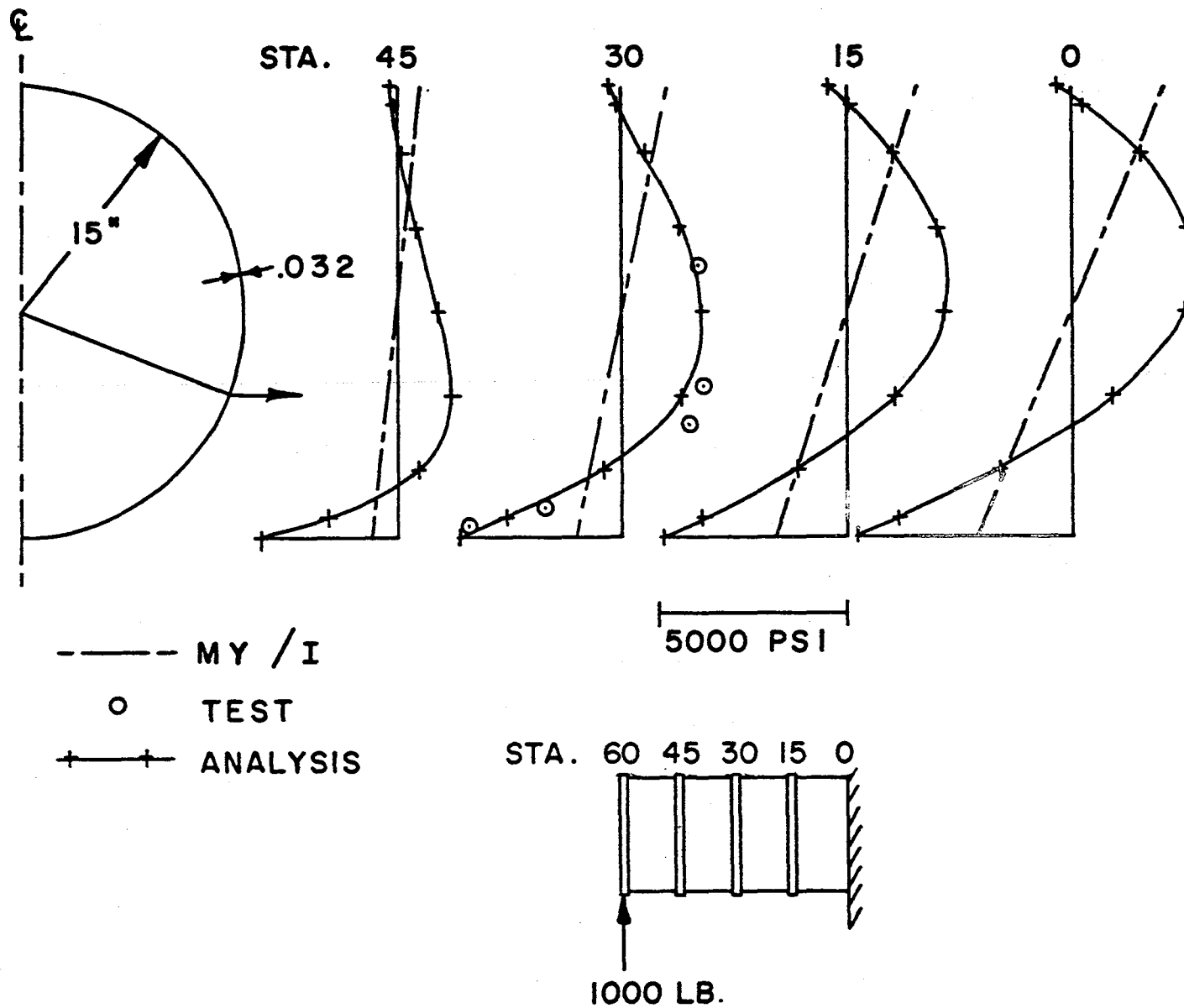


Fig. 1 Axial stress in a cylindrical Shell

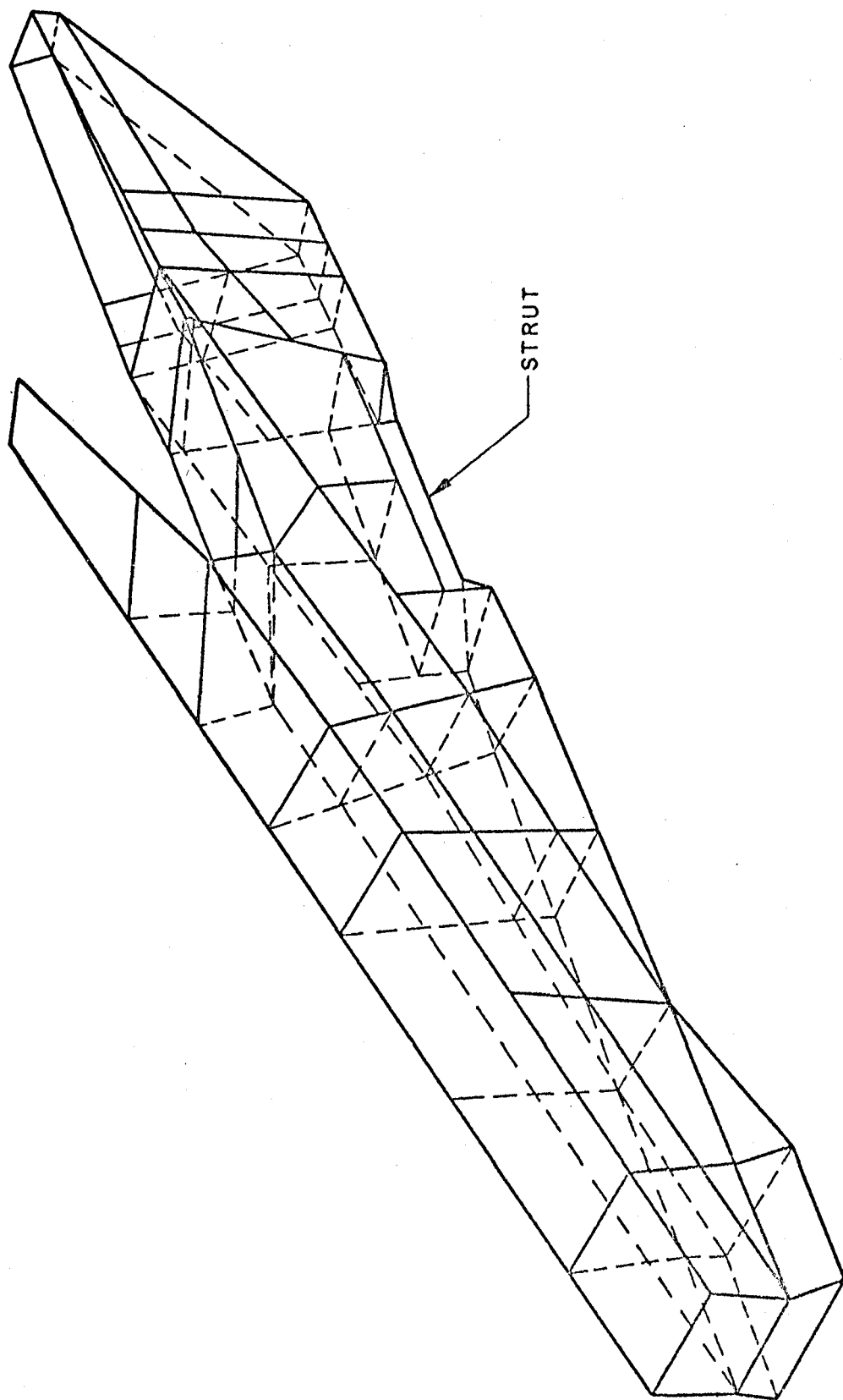


Fig. 2a Idealized pylon structure

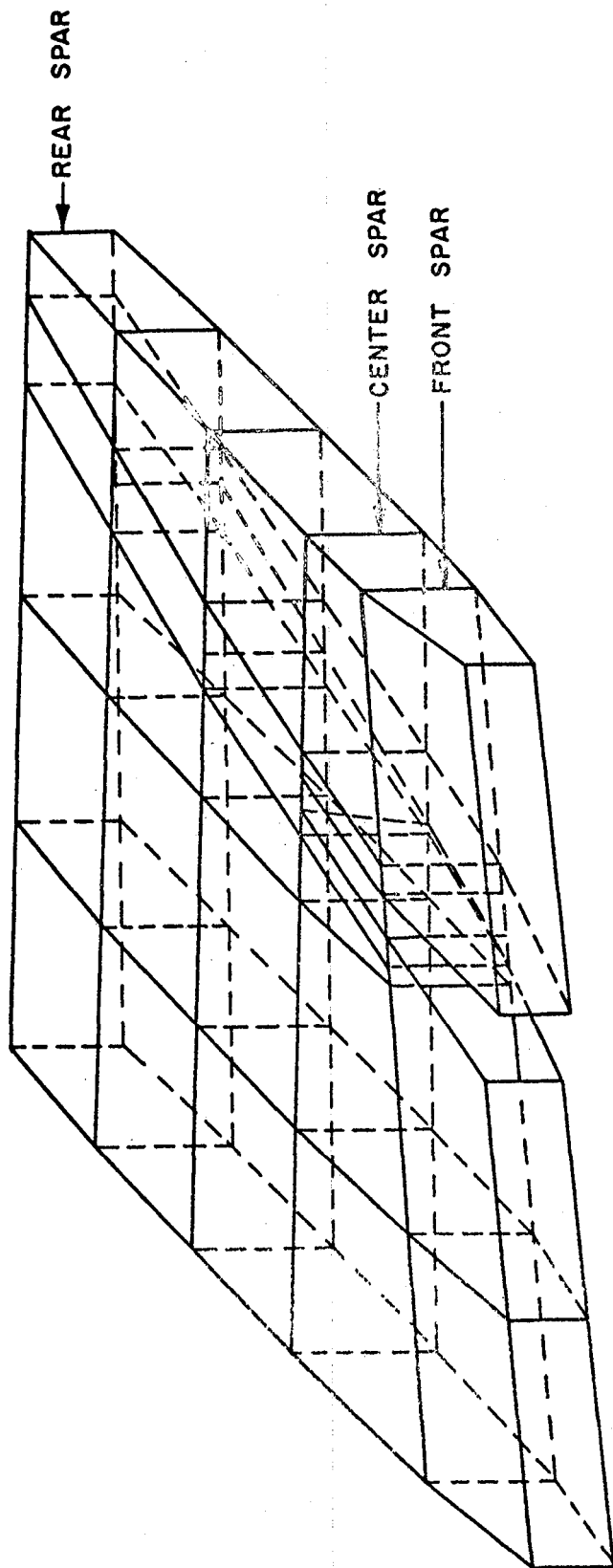


Fig. 2b Idealized wing structure



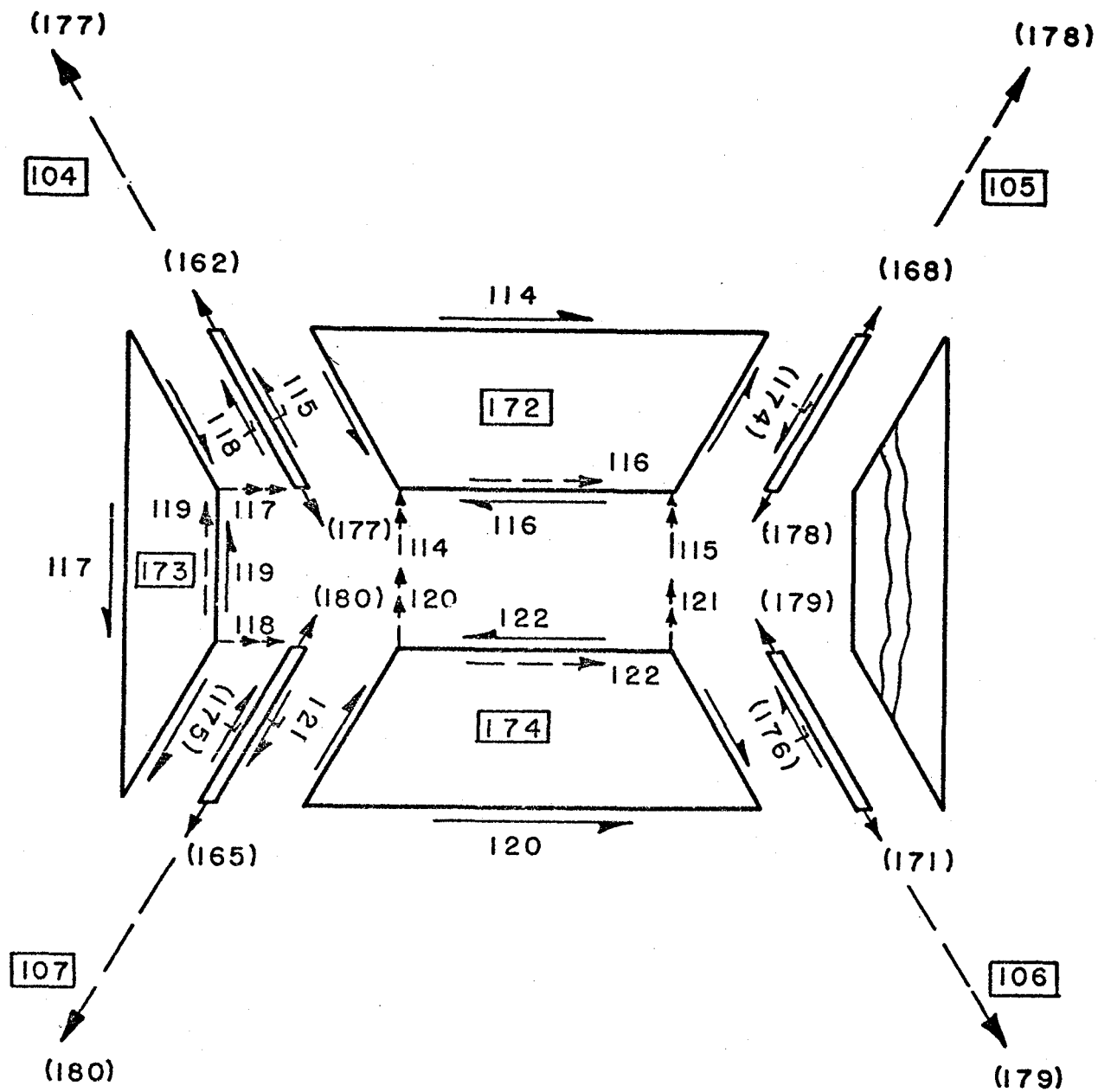


Fig. 3 Typical free body diagram

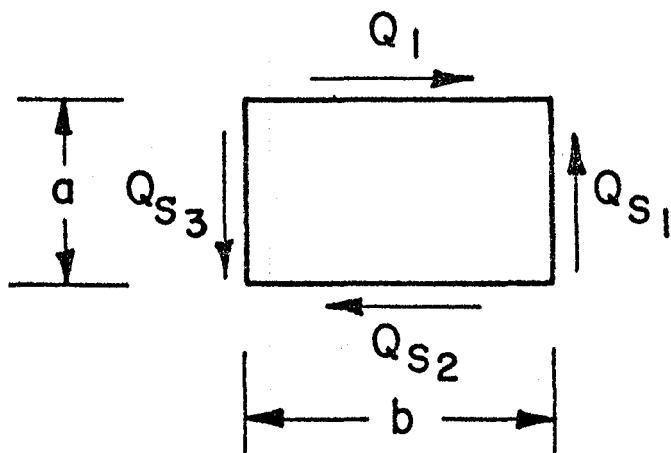


Fig. 4 Principal and subordinate forces

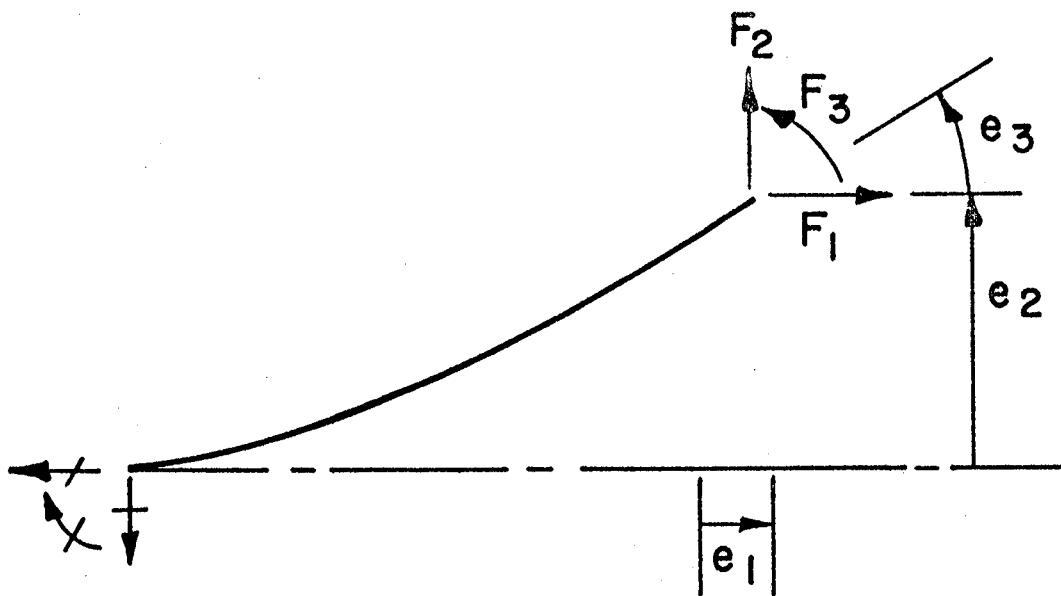


Fig. 5 Element forces and deformations

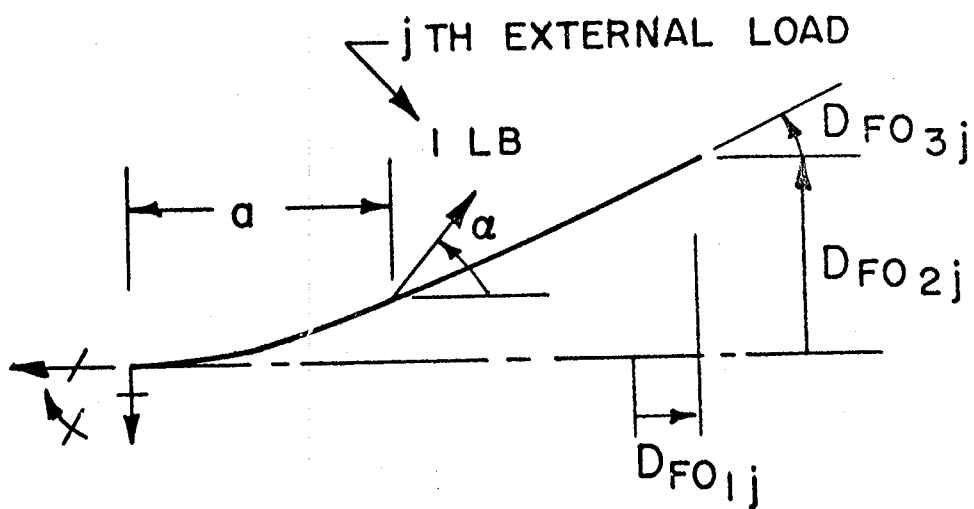


Figure 6

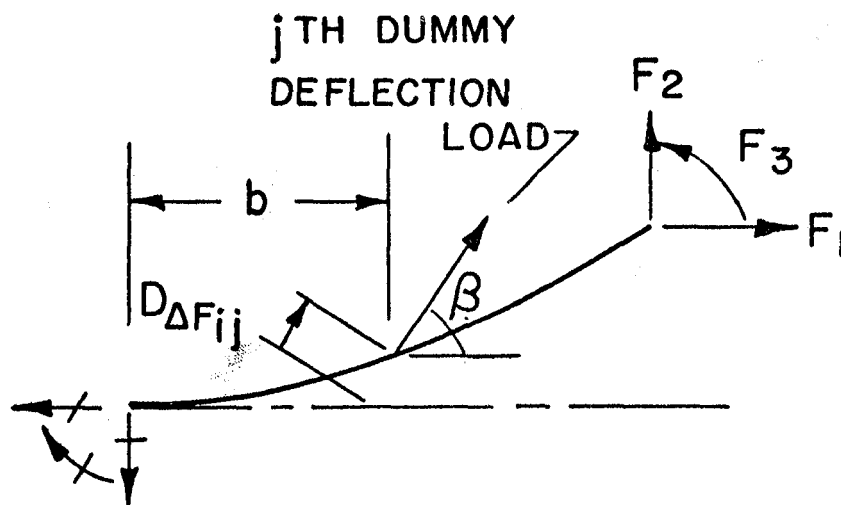


Figure 7

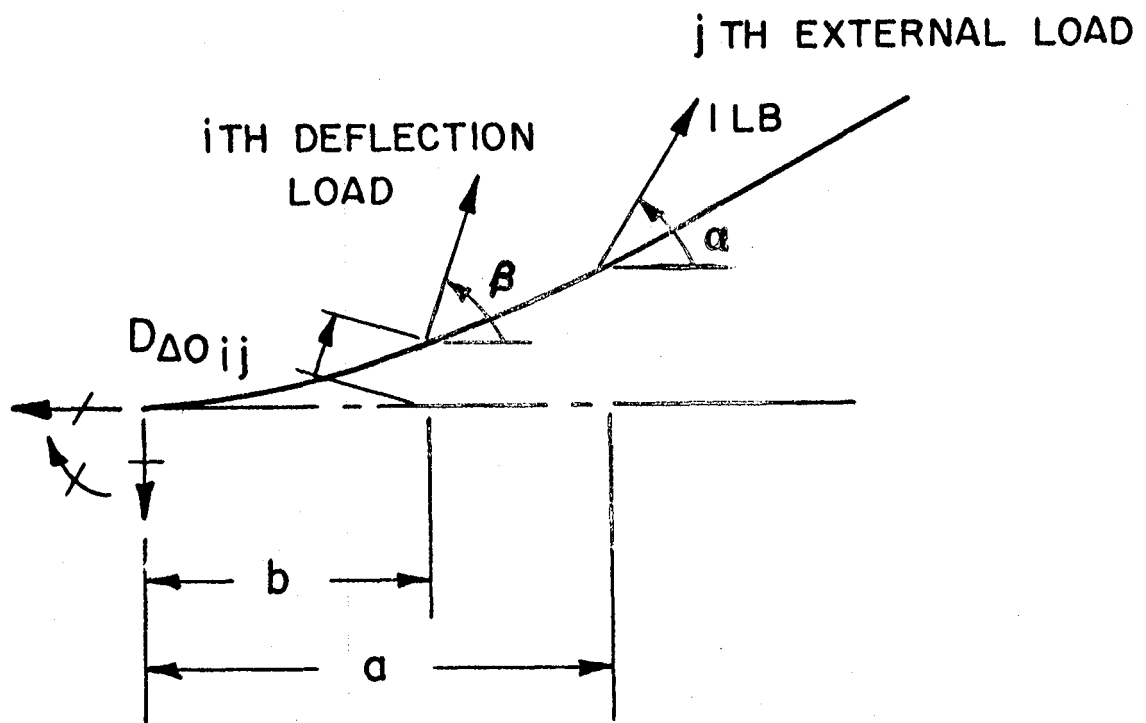


Figure 8

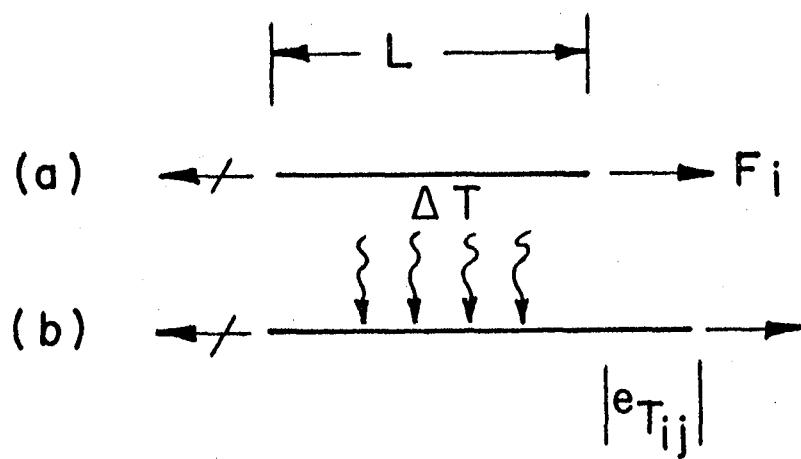


Figure 9

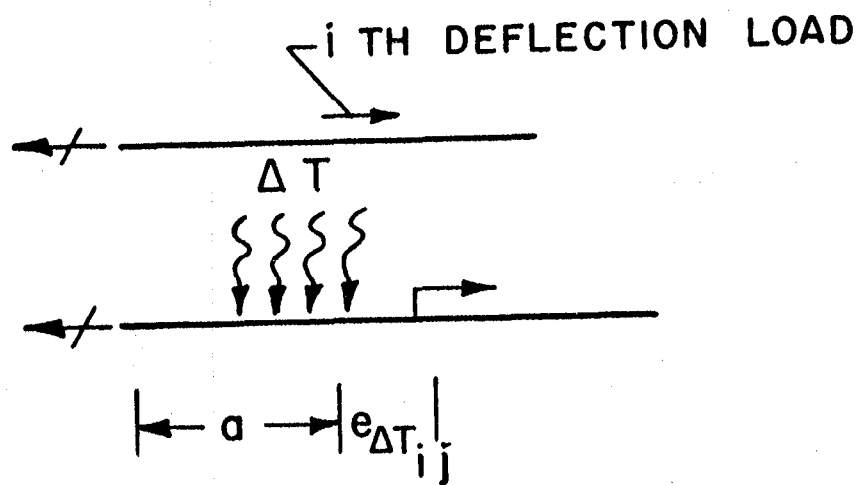


Figure 10



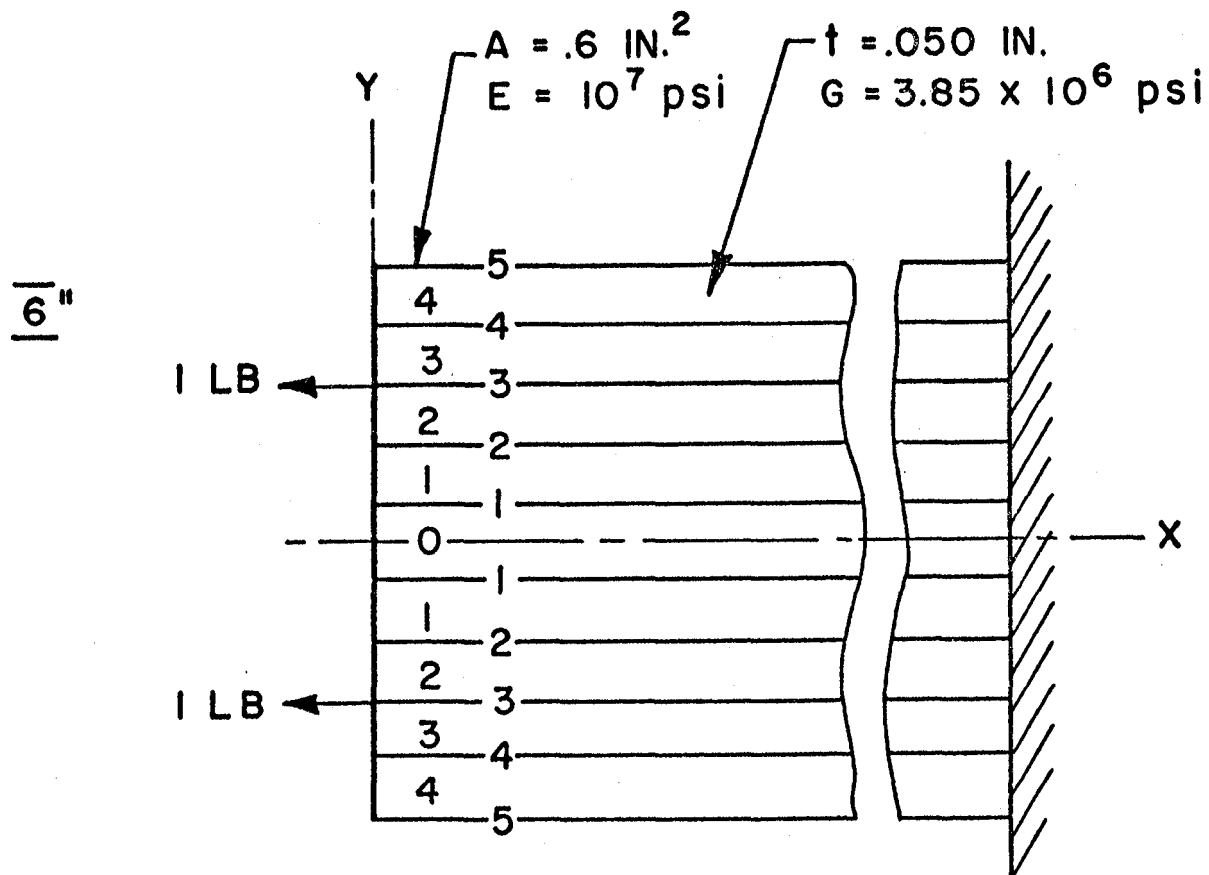


Fig. 11 Shear lag problem

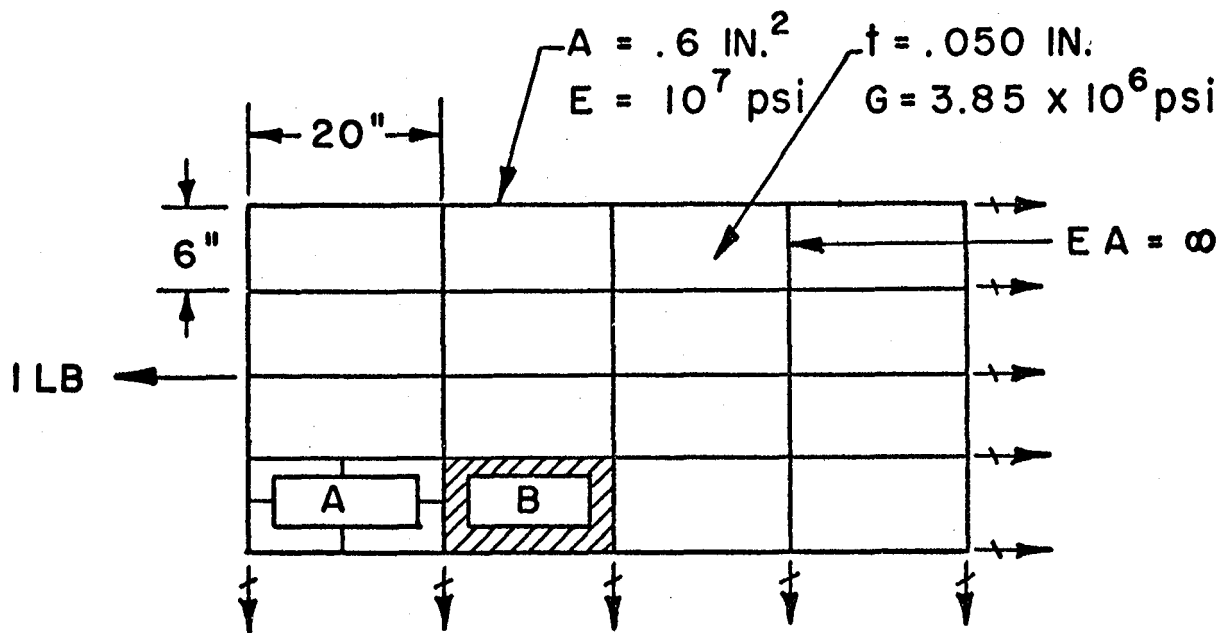


Fig. 12 Idealized structure for shear lag problem

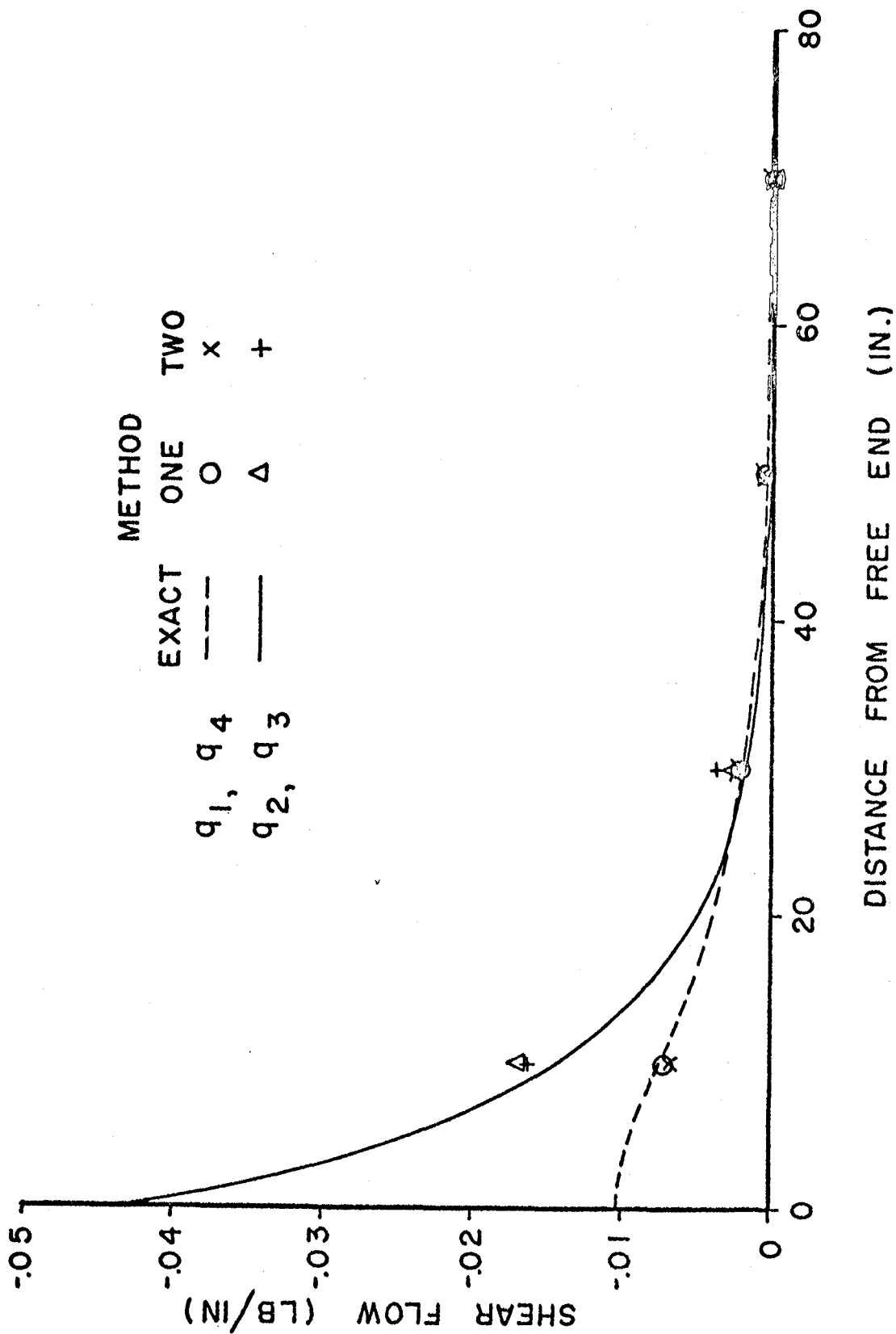


Fig. 13 Comparison of shear flows

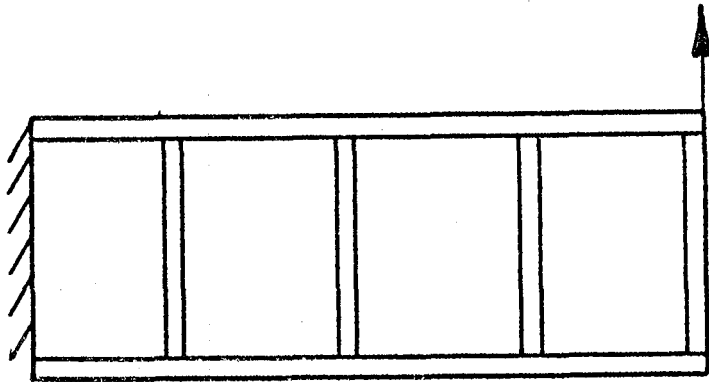


Figure 14

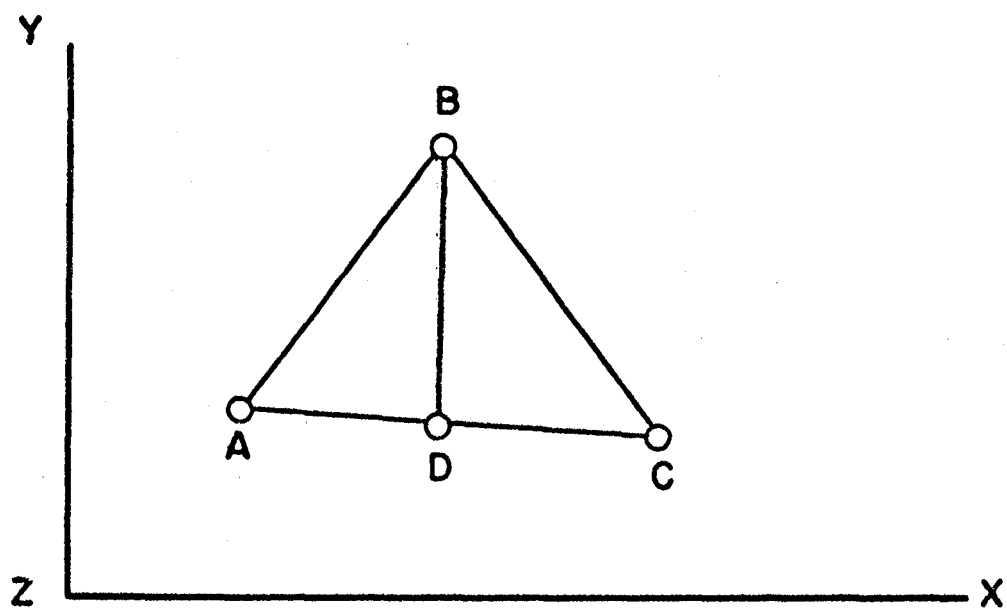
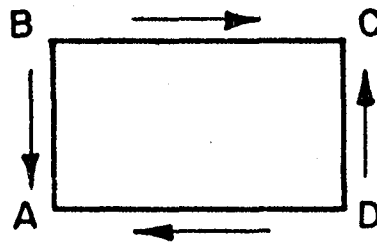
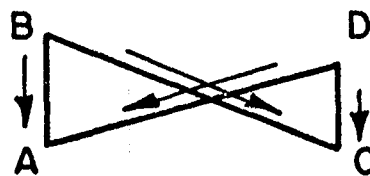


Fig. 15      Defining and derived coordinates



PLAN VIEW



EDGE VIEW

Fig. 16 Warped shear panel

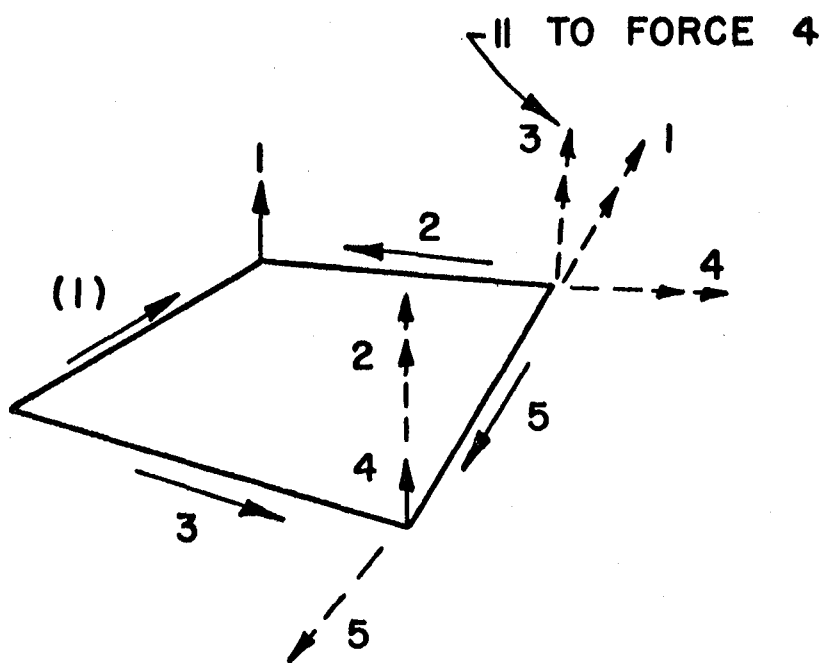


Fig. 17 Equilibrium of a warped shear panel

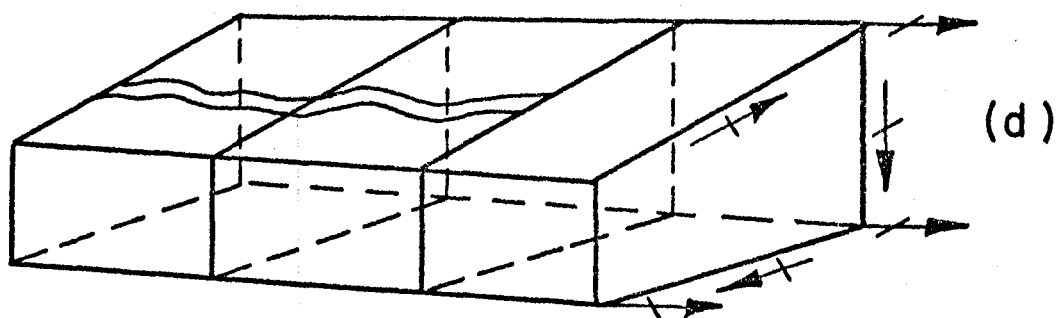
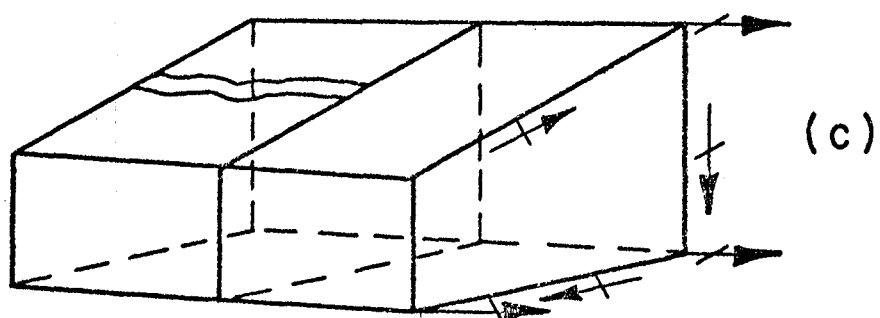
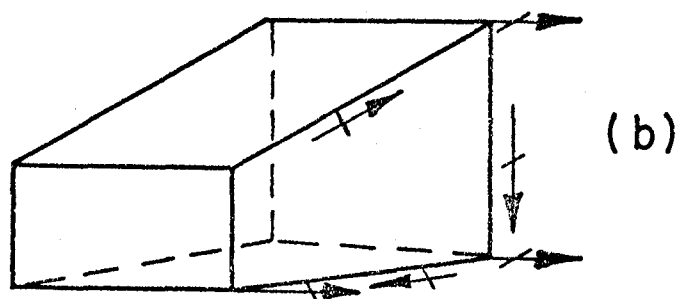
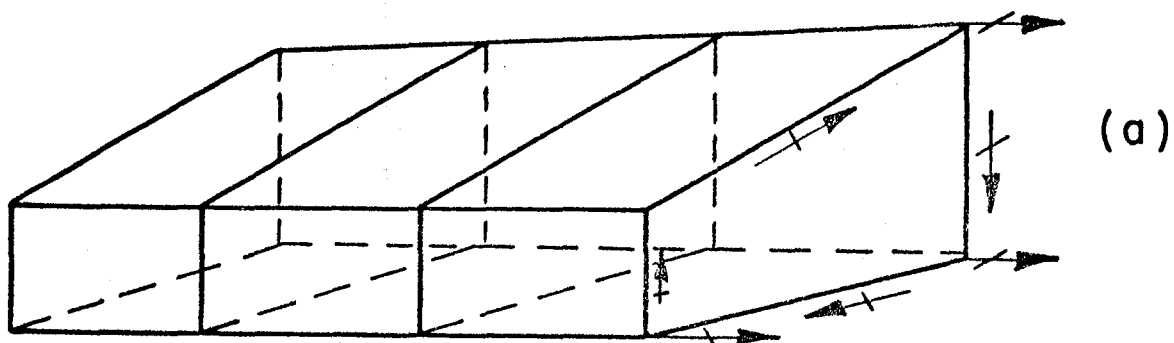


Fig. 18 Cutting a structure by the building method



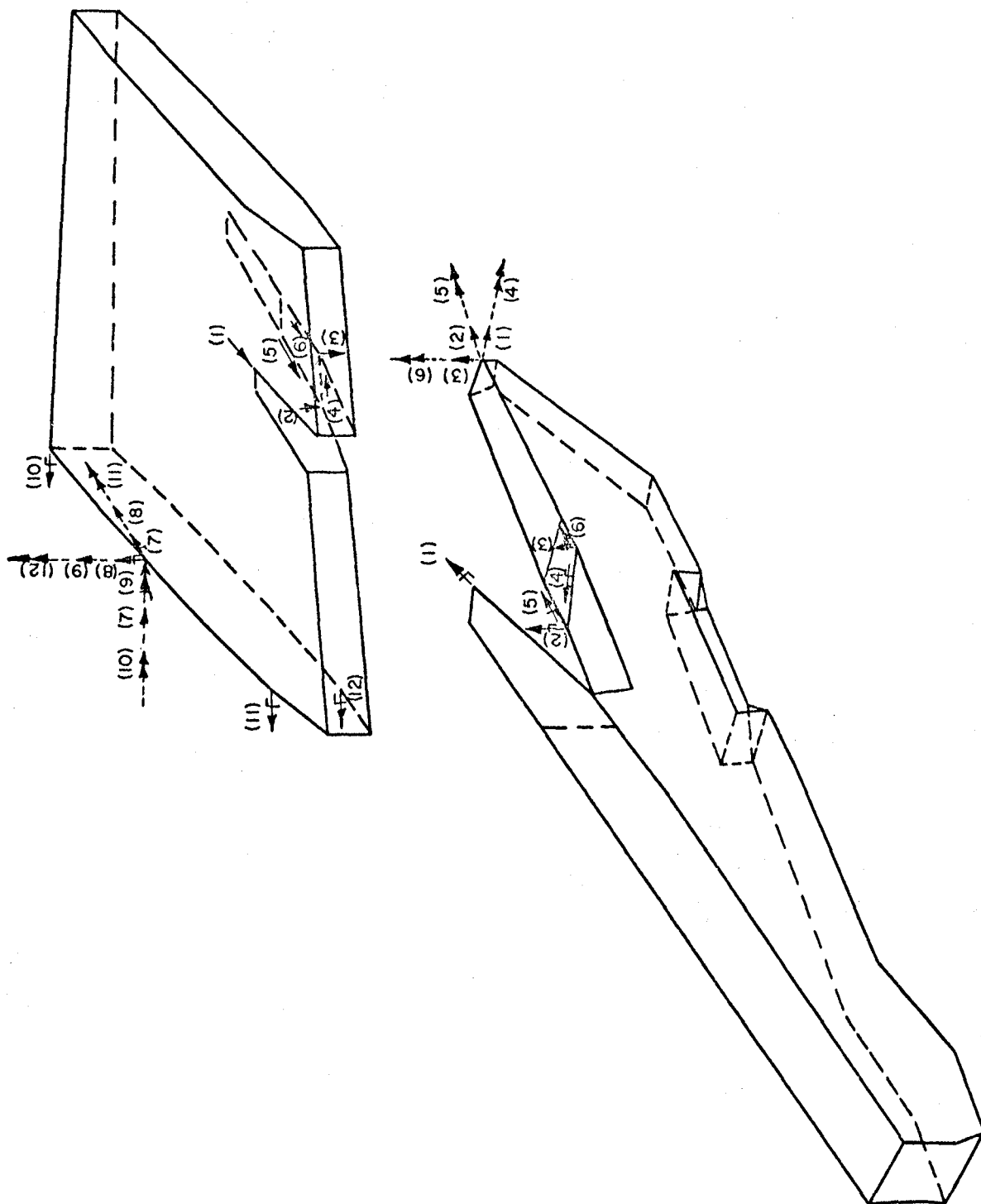


Fig. 19 Free body diagram for joining structures

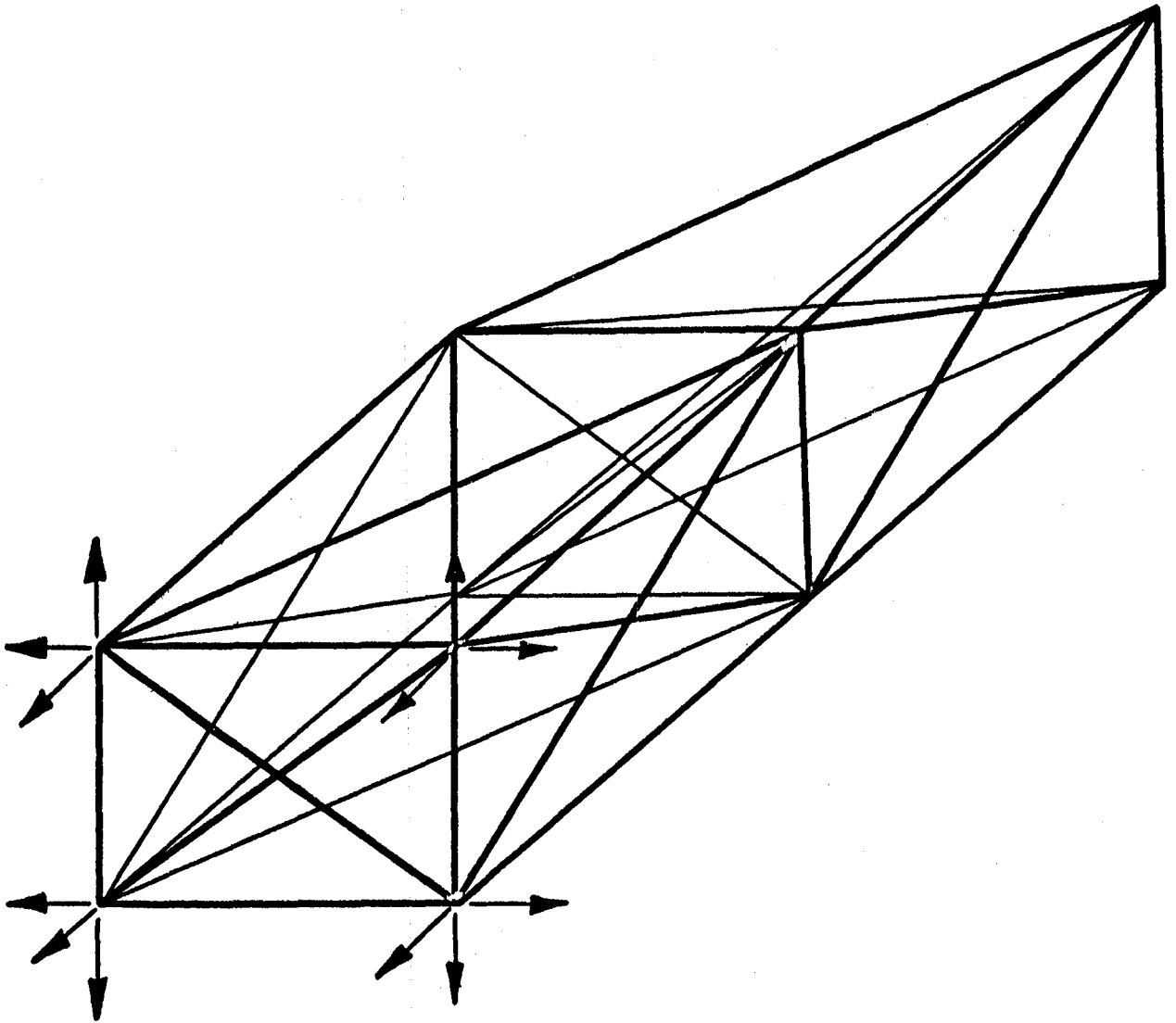


Fig. 20a      Statically indeterminate structure

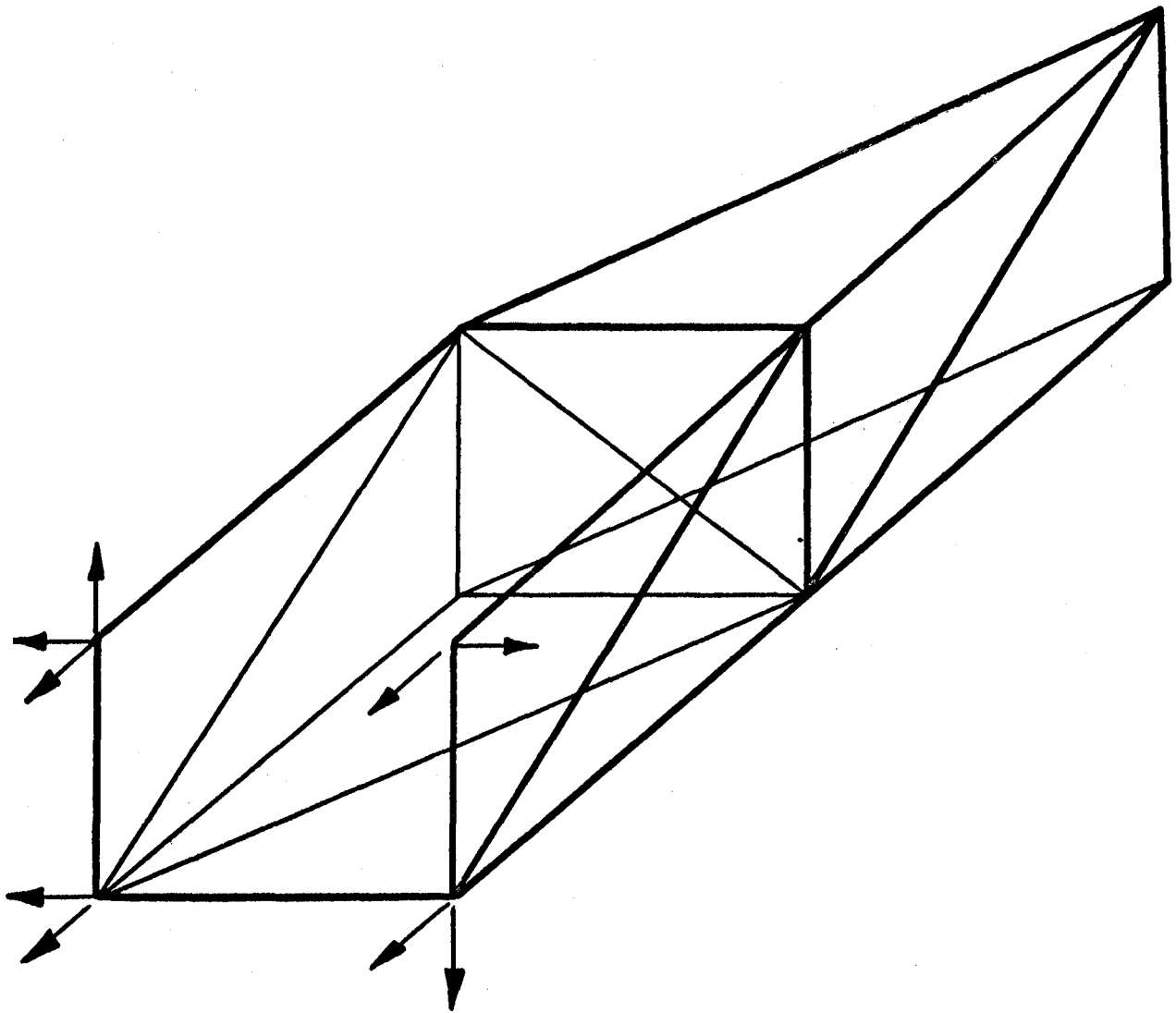


Fig. 20b Machine cut structure

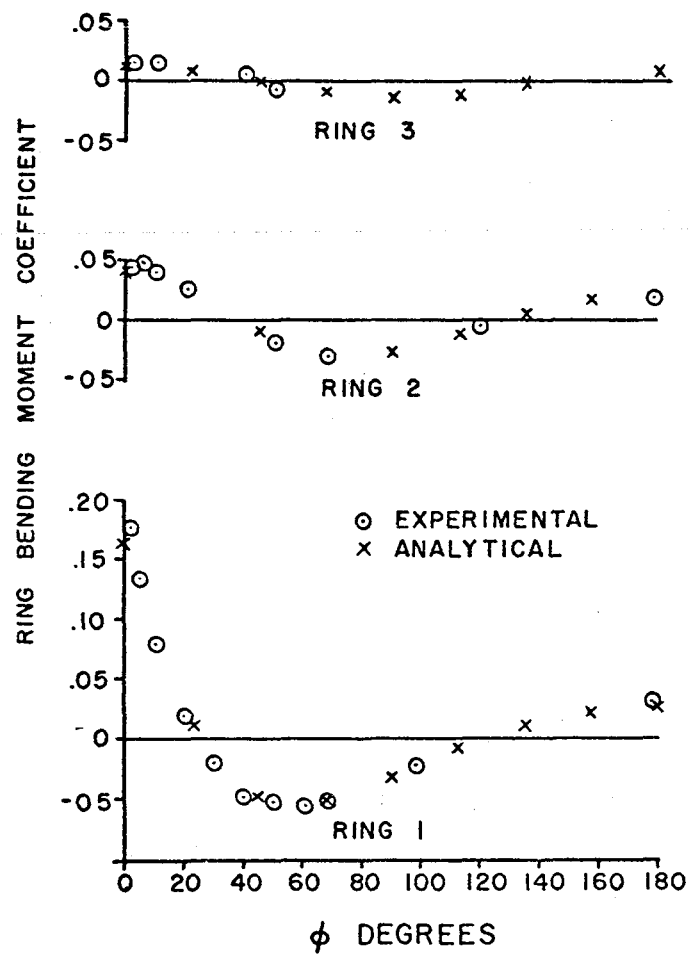
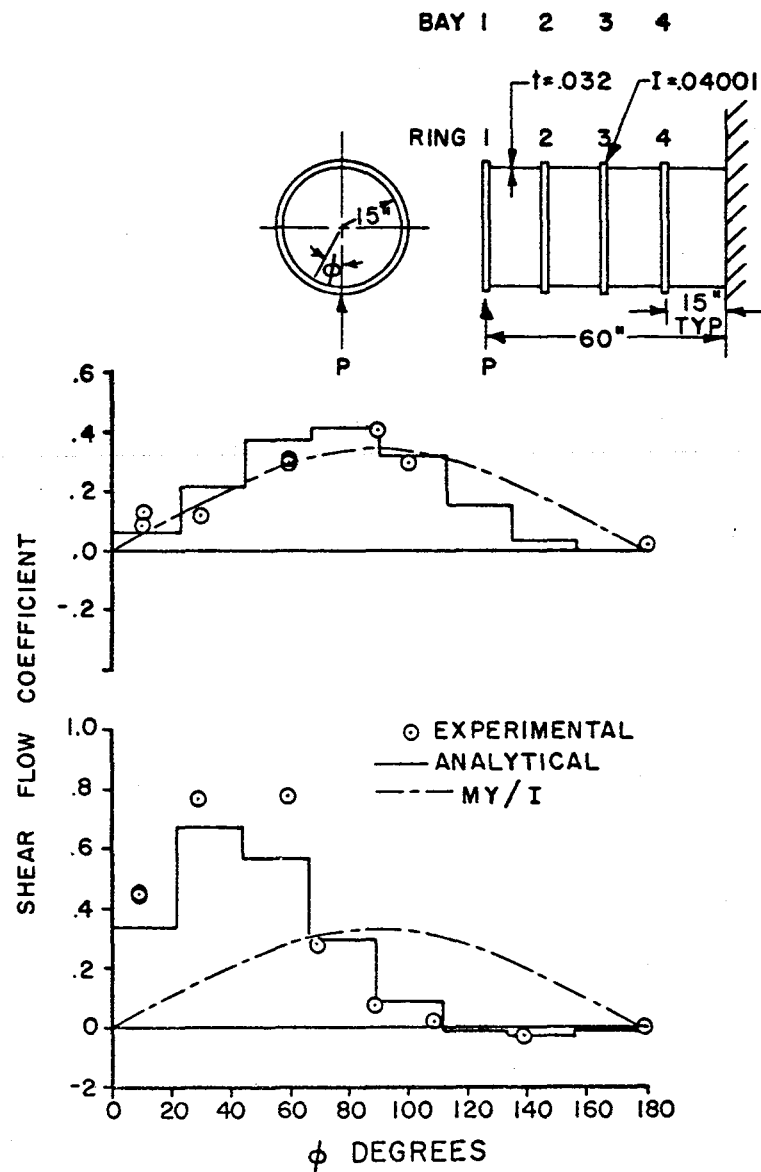


Fig. 21

Frame bending moment and skin shear flow



# AXIAL STRESS - BENDING

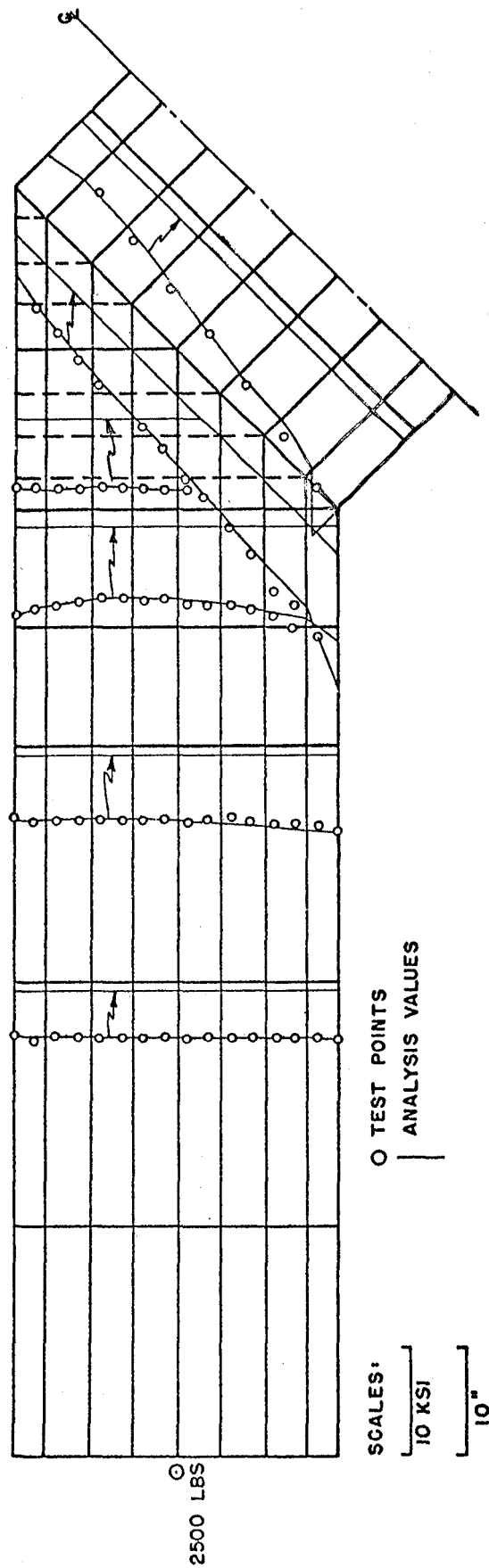


Fig. 22 NACA swept box

# SHEAR STRESS-BENDING

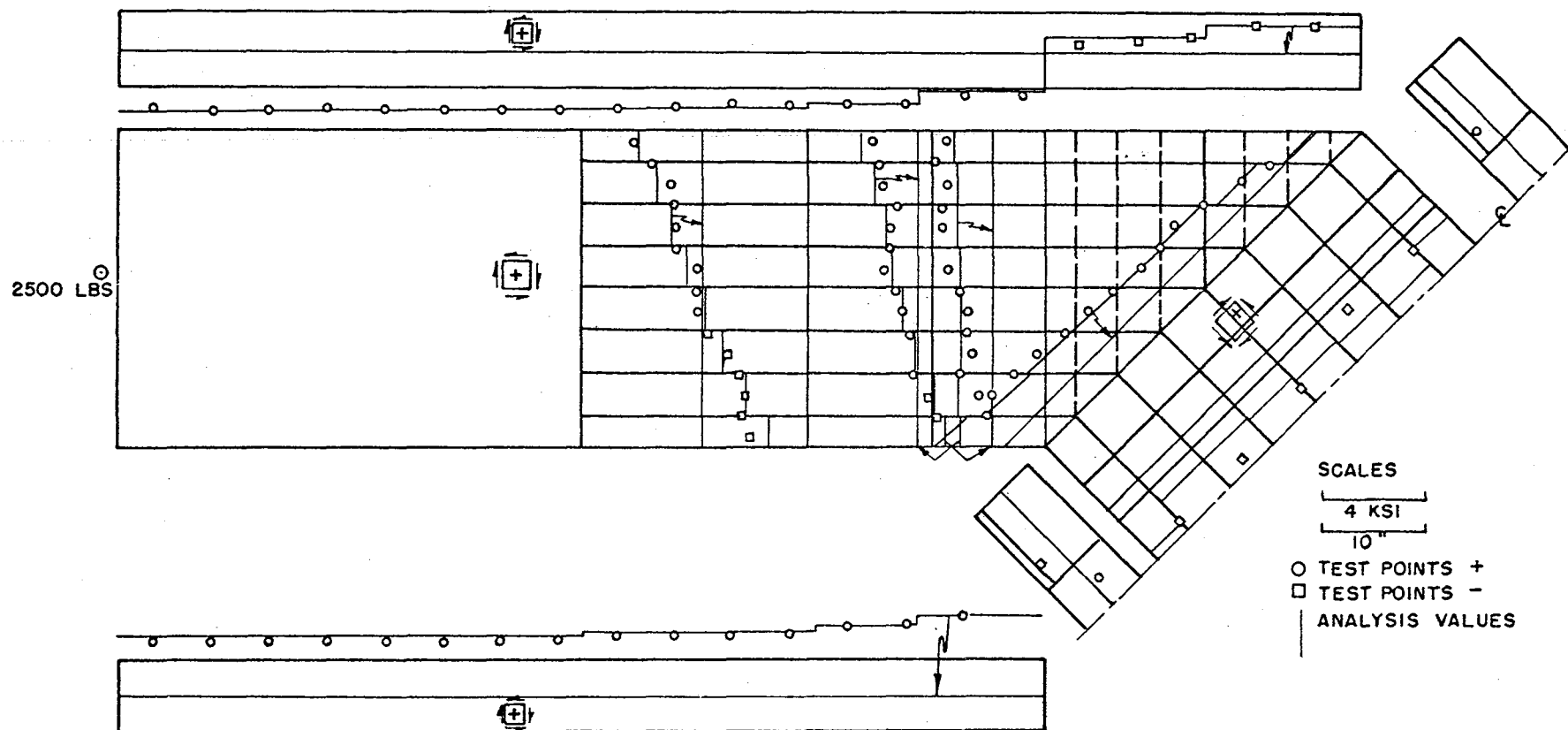


Fig. 23 NACA swept box

# AXIAL STRESS-TORQUE

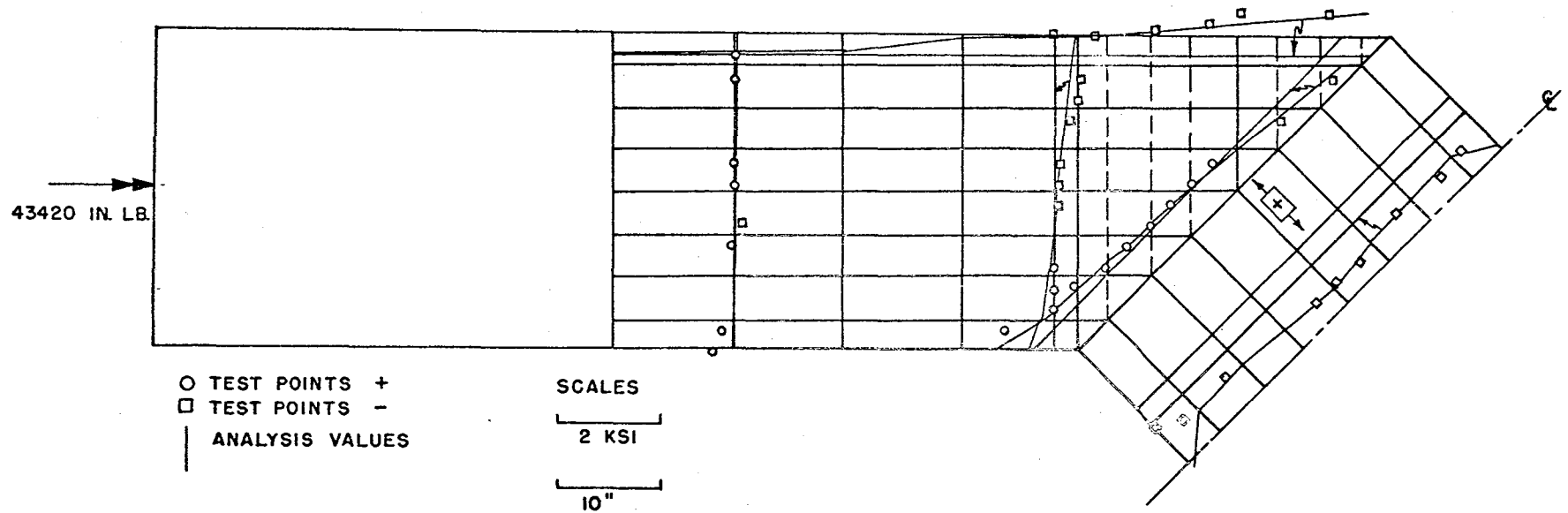


Fig. 24 NACA swept box

# SHEAR STRESS - TORQUE

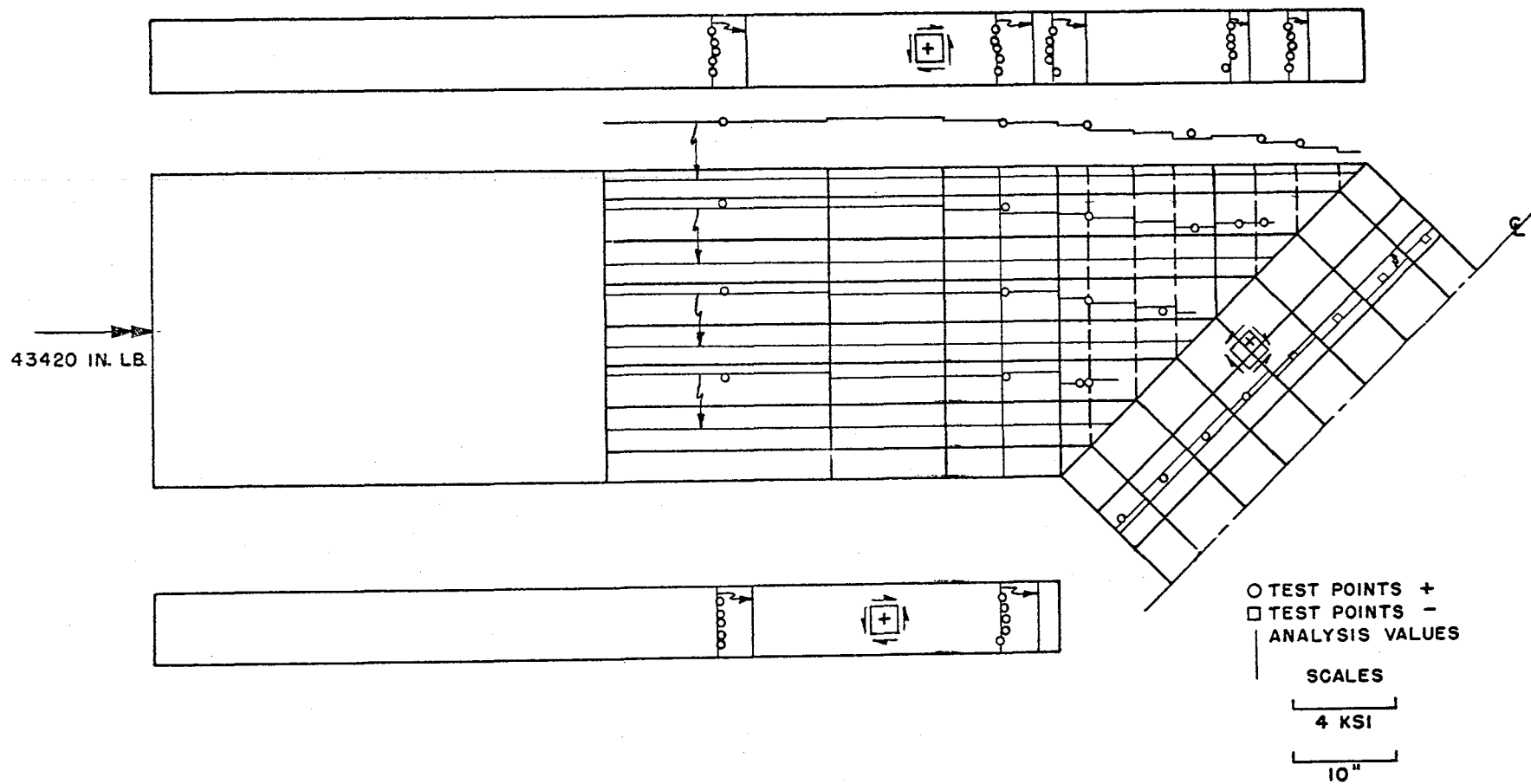
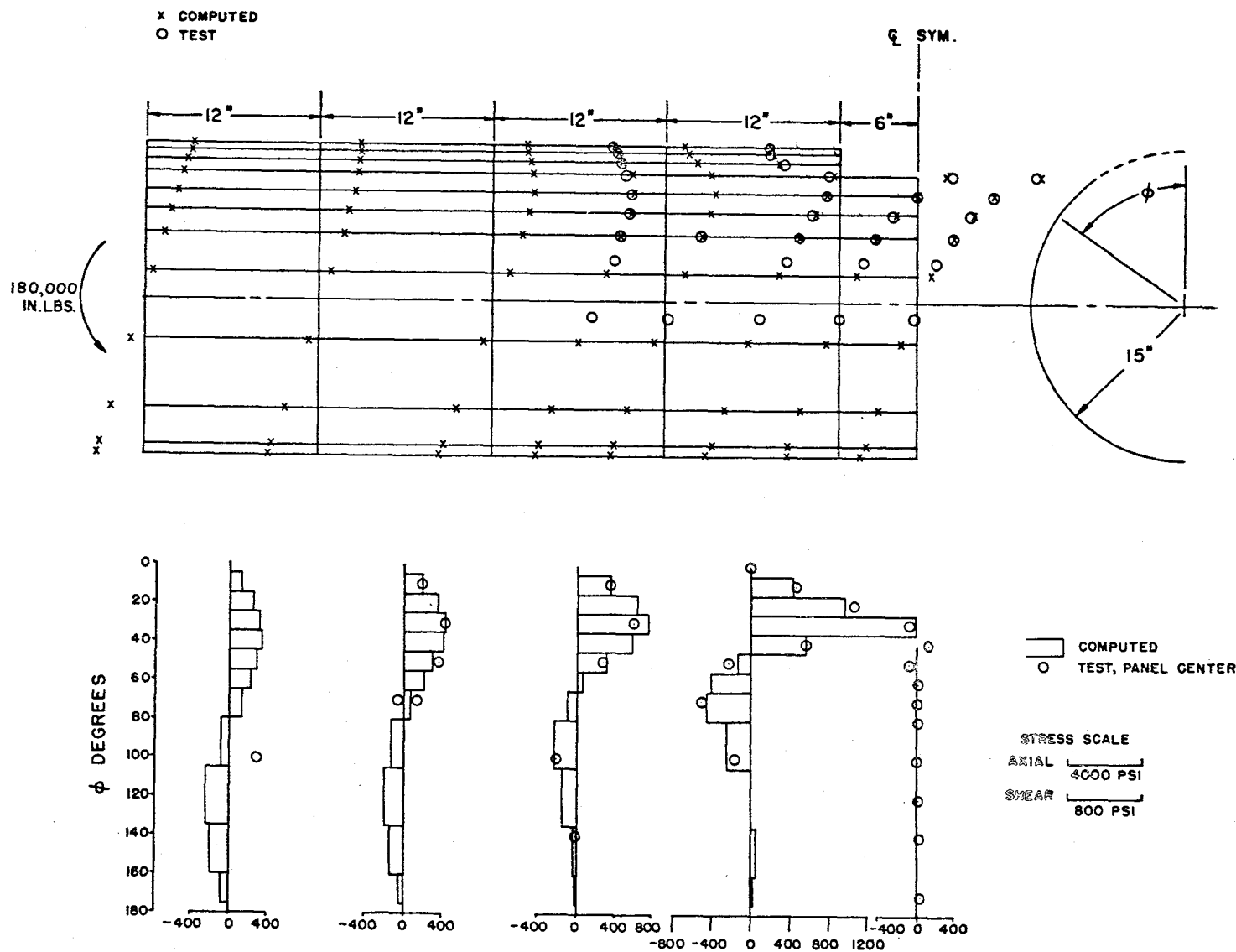


Fig. 25 NACA swept box

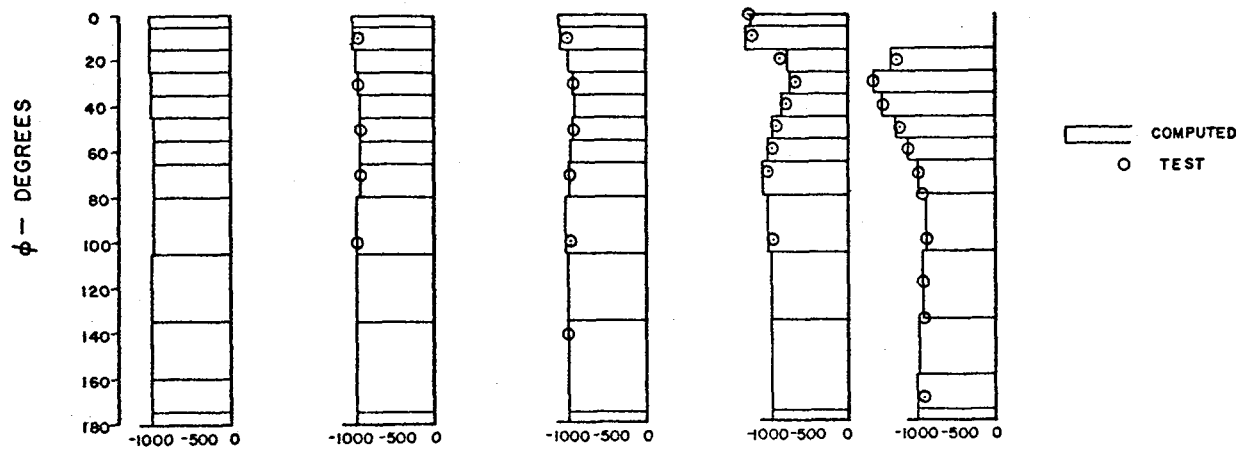
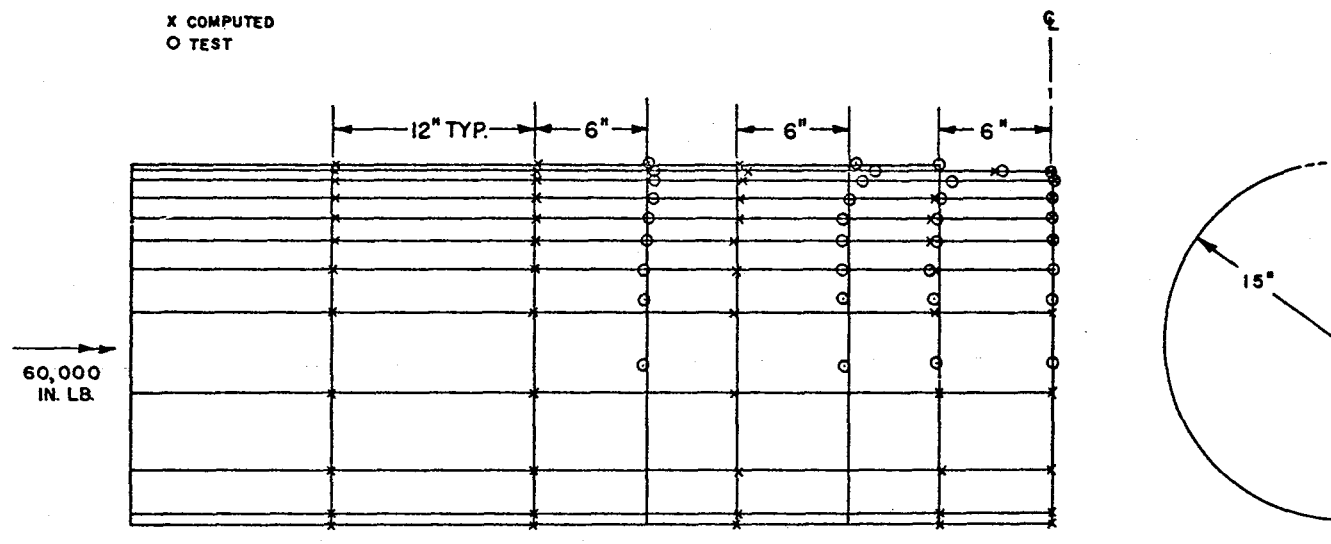


# CYLINDER WITH CUTOUT-BENDING



SHEAR STRESSES-PS.I.  
Fig. 26 NACA cutout cylinder

# CYLINDER WITH CUTOUT-TORQUE



SHEAR STRESSES - psi

Fig. 27 NACA cutout cylinder

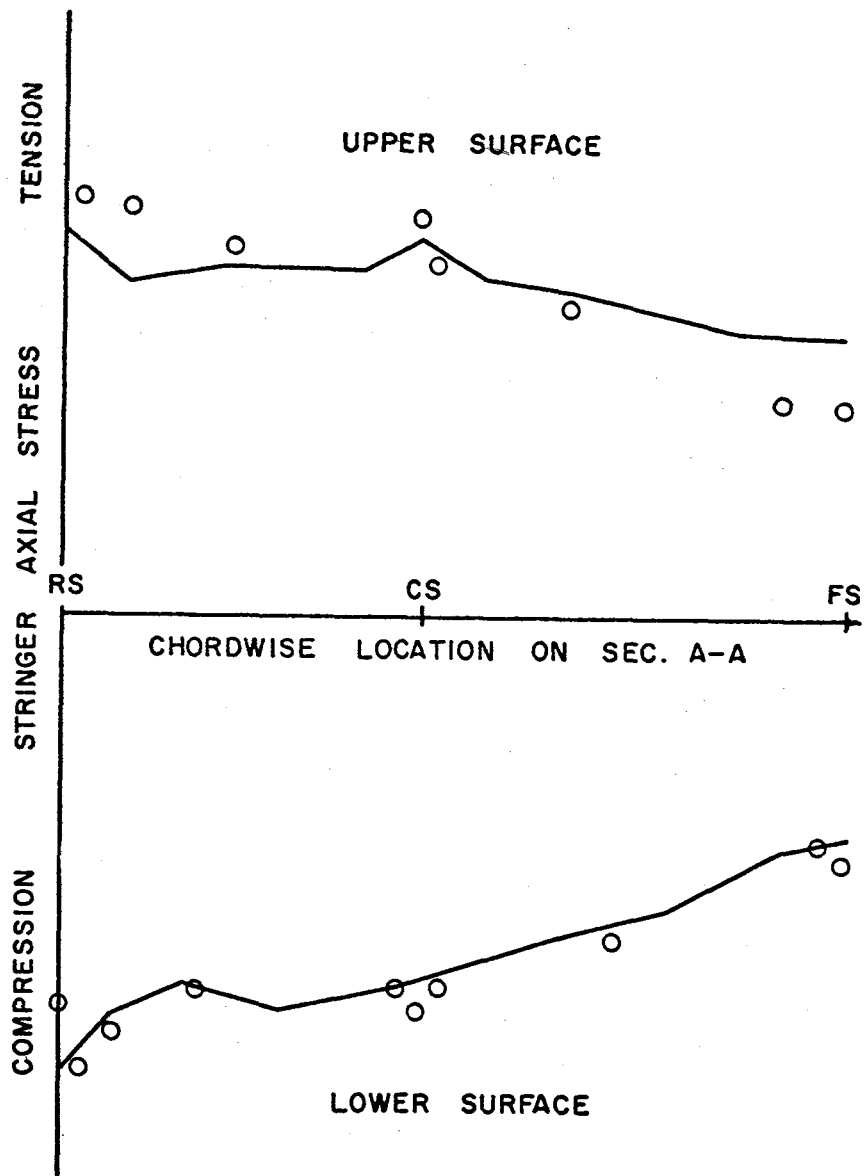
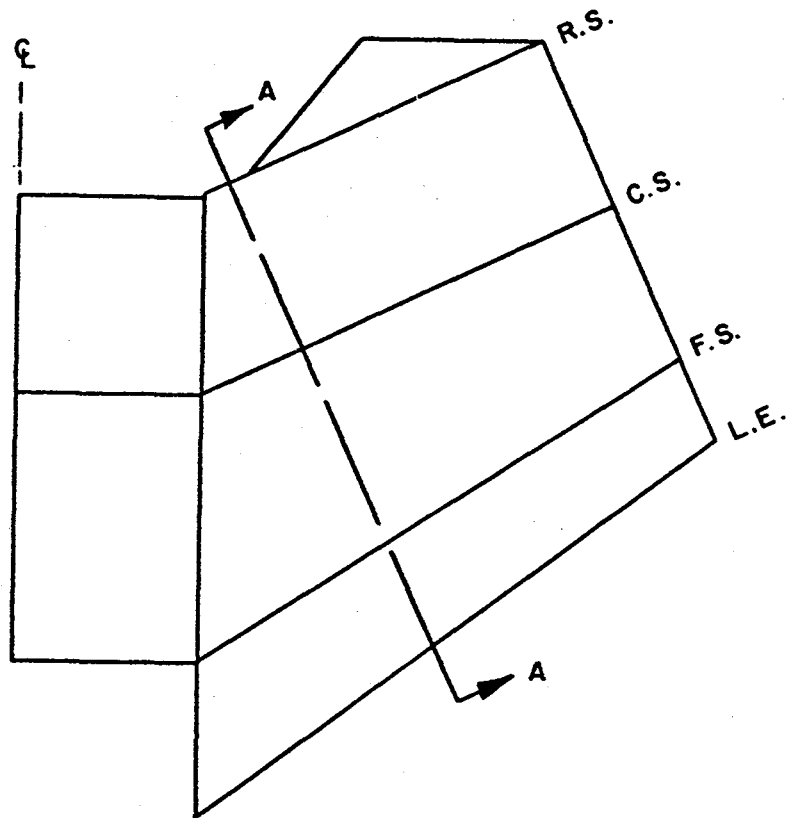


Fig. 28 Axial stress in swept wing structure

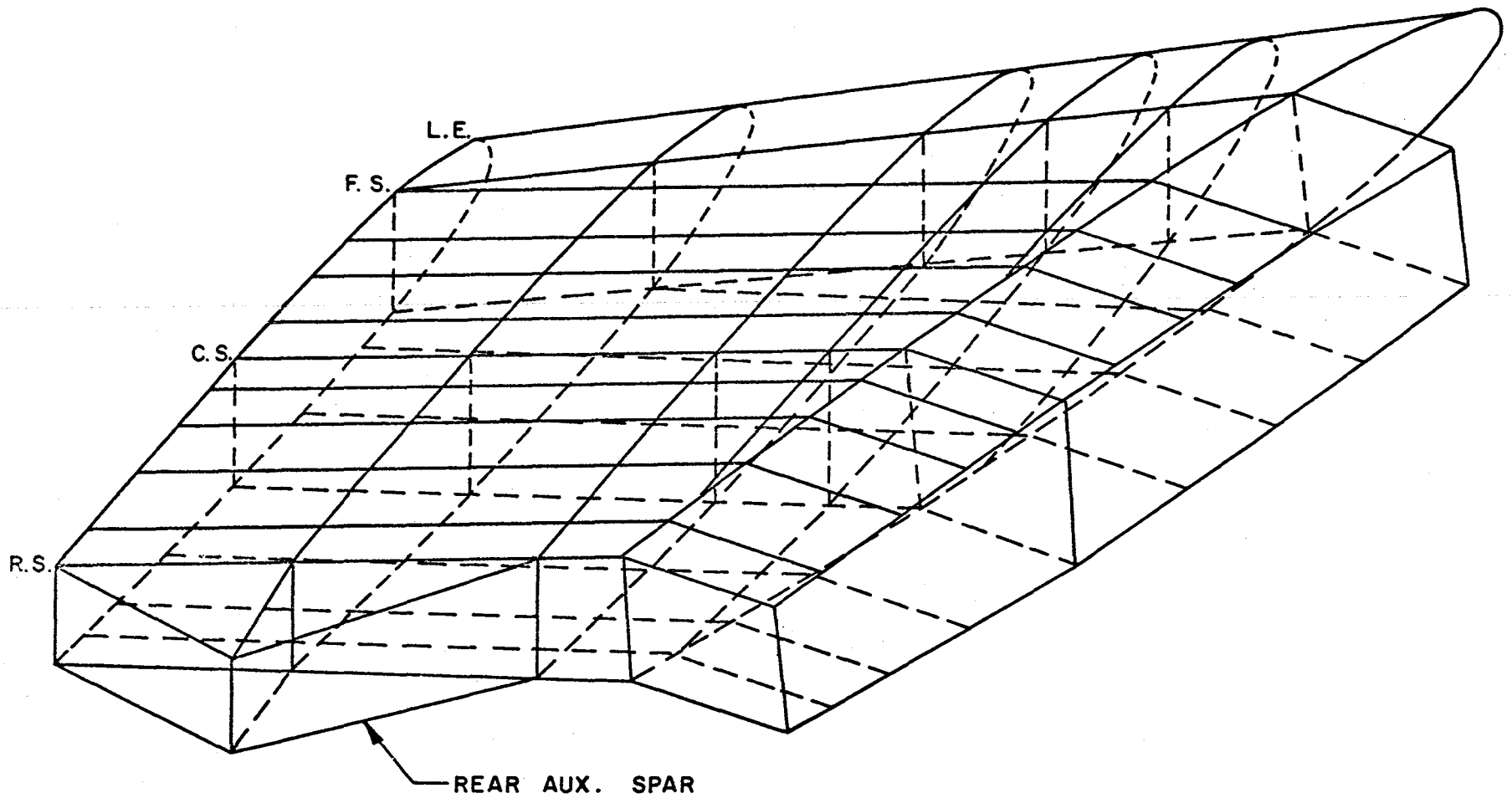


Fig. 29 Swept wing root idealized structure

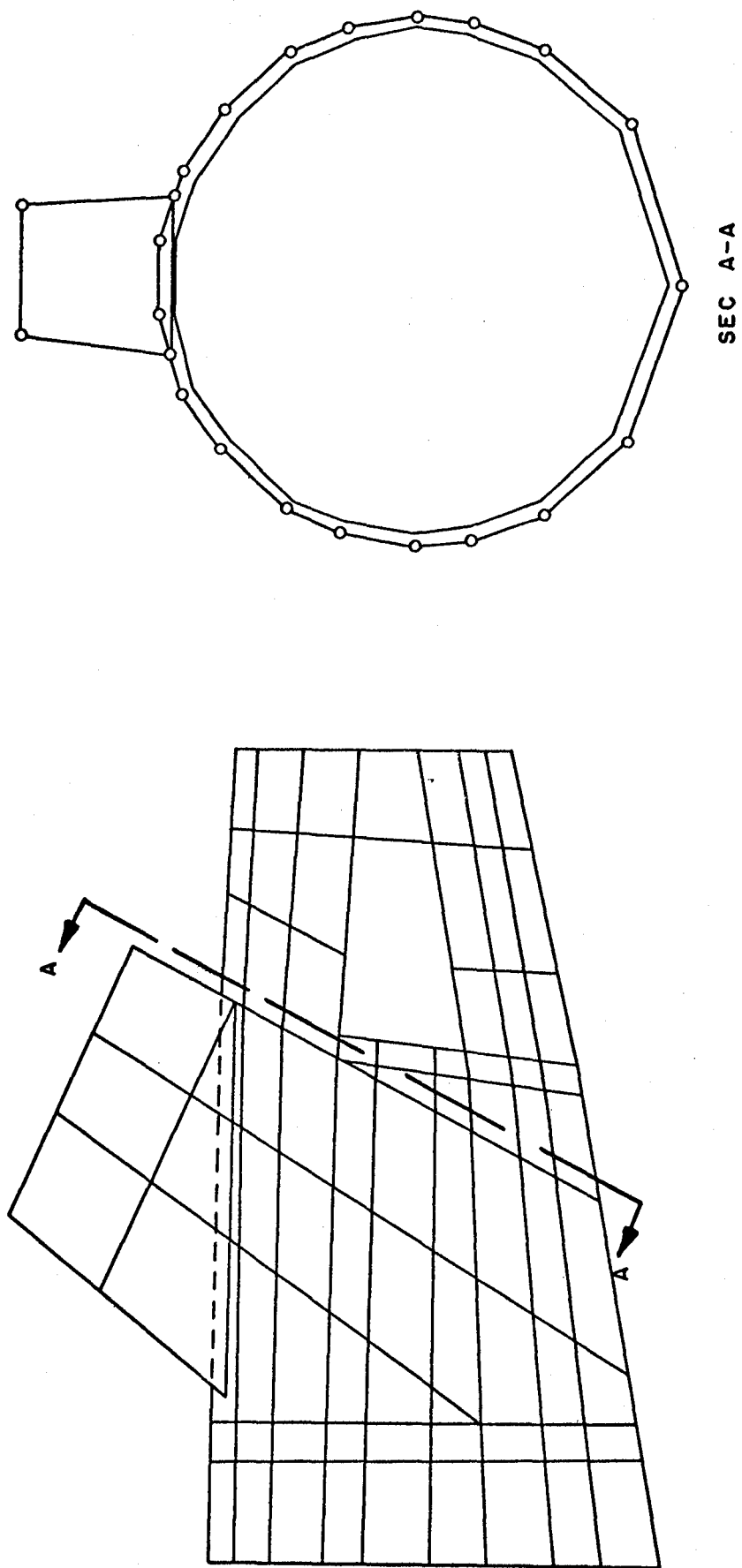


Fig. 30 Fuselage tail section idealized structure

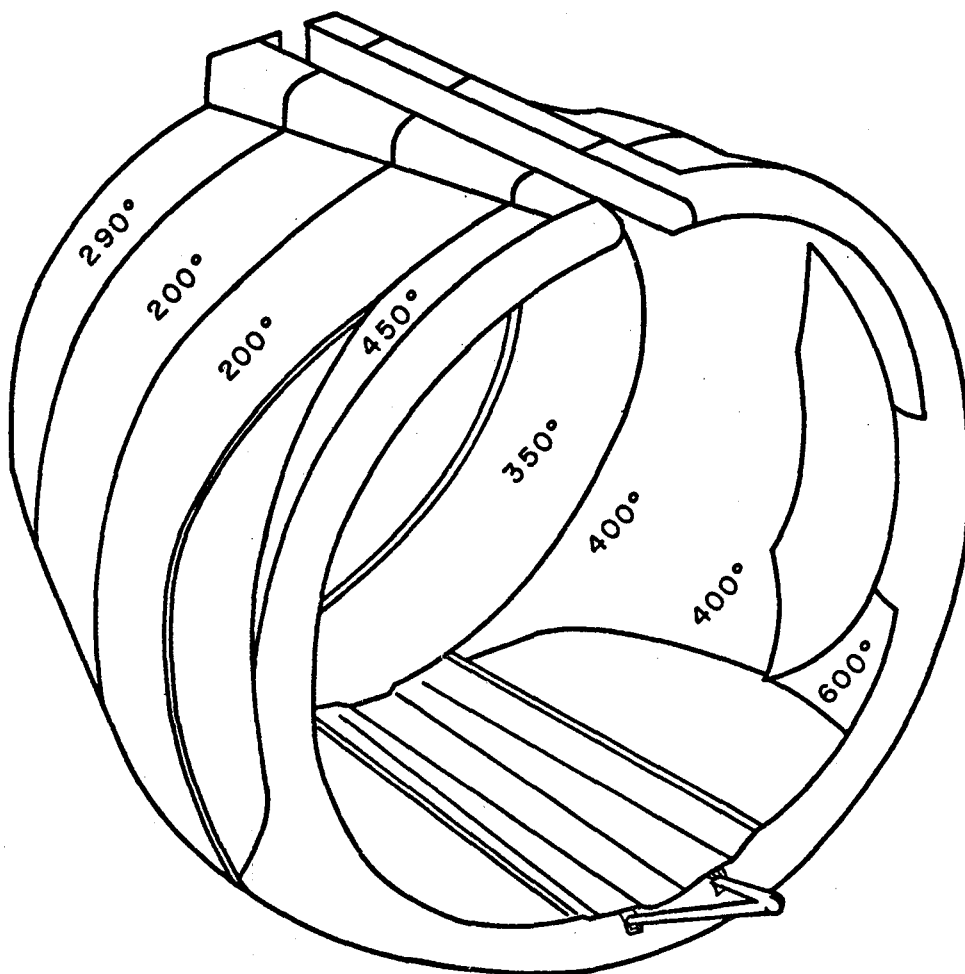


Fig. 31 Ejector-reversor

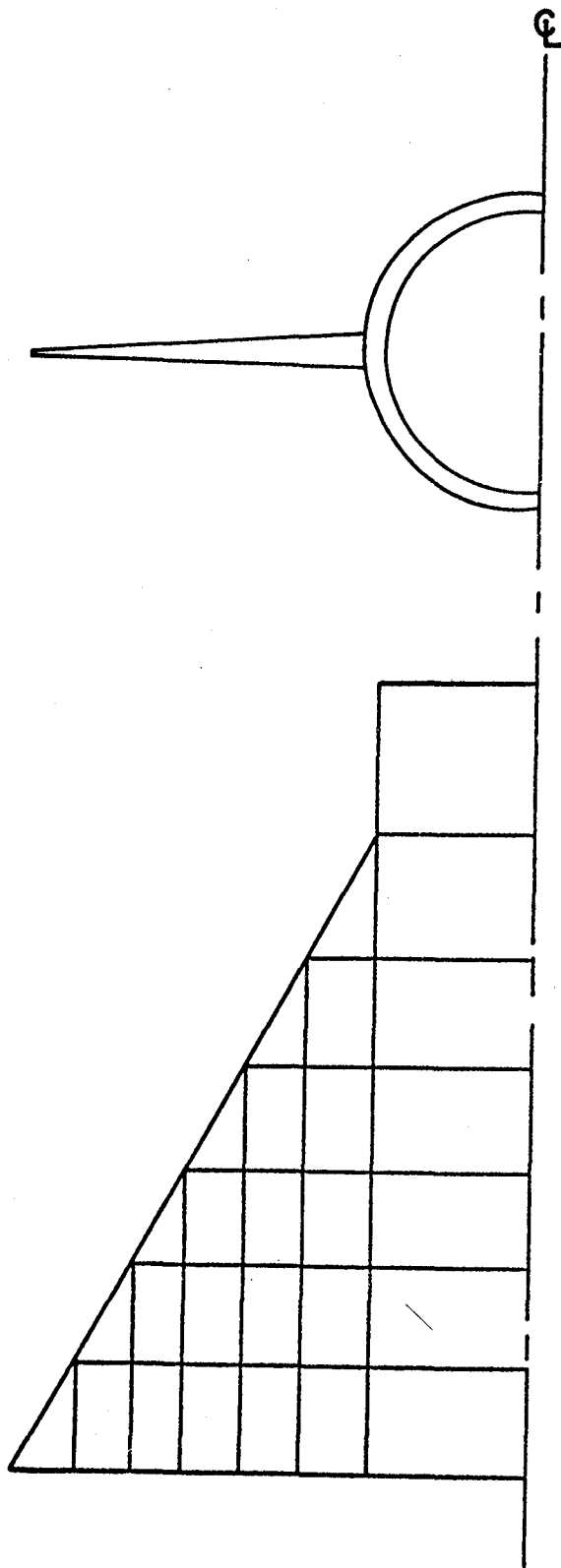


Fig. 32 Low aspect ratio wing idealized structure